TEACHERS’ MATHEMATICAL CONTENT KNOWLEDGE AND STUDENTS’ PROGRESSION IN LEARNING OF FRACTION AND PROPORTION

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Abstract

Proportional reasoning causes considerable difficulty for students. One reason for this is, that a lack of basic understanding of fractions in earlier years, causes difficulties in the middle years. Moreover, learning of fractions and proportion is a long-term process and students encounter it continuously from grade 1 to 9. It is also difficult to teach, and teachers’ mathematical content knowledge plays a crucial role in students’ learning of mathematics. The purpose of this study is to analyse the effects of teaching, and the influence of teachers’ mathematical content knowledge about fractions and proportion on students learning in progression from grade 4 and 5 to grade 8. The method contains pre- and post-tests for 86 students, classroom observations, checking students’ written solutions and interviews of 35 selected students after the post-test. The intervention includes construction of educational materials (EM), teachers participating in seminars-training related to EM and teacher’s implementation of EM. The EM contains tasks for students and a teachers’ guide with aims and goals for teaching and a theoretical background. Tools for analysis were a methodological design, *Variation theory*, and a theoretical approach, *Mathematical Content Knowledge for Teaching*. In focus of the analysis was students learning in progression, related to variation and crucial aspects of learning. Findings from this study shows that teachers’ mathematical content knowledge and their ability to identify the objects of learning and apply this in teaching is very important for students’ learning and progression in their learning. Most students showed an ability to learn, but their performance was intimately linked to teachers’ perception of the crucial aspects in teaching, and variation. Moreover, anomalies in students’ perceptions of basic concepts caused obstacles in their learning. Some anomalies seem to have followed students from middle School to grade 8. Finally, the study illustrated how anomalies arise if misconceptions are not noticed by teachers. The outcome of the study can explain more about crucial steps in teaching and learning of fraction and proportion. The study pay attention to challenges in mathematics teaching.

Keywords: *Mathematical content knowledge, fraction and proportion, variation, progression and learning.*

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1. Introduction

Many of the problems that students are supposed to solve, from middle school and on, are related to ratio and proportion. At the same time proportional reasoning causes considerable difficulty for the students (Dole & Shield, 2008). One reason for this is, that lack of basic understanding of ratio in early years may cause difficulties in the middle years and lack of ratio knowledge in adult life (Behr et al., 1992). According to Ohlsson (1988) proportion, ratio and fractions are connected to each other. At the same, learning of fractions is complicated because its “many related but only partly overlapping ideas” (p. 53). To sum up, fractions are a difficult to learn and learning of ratio and proportion is a long-term process that assumes a continuity from grade 1 to 9 (Hackenberg & Lee, 2015). As different aspects of fraction and proportion are introduced during different school years, usually by different teachers, there is an urgent need of a long-term planning to secure a continuity in teaching and learning. “In general, the researchers found that teachers with a relatively weak conceptual knowledge of mathematics tended to demonstrate a procedure and then give students opportunities to practice it. Not surprisingly, these teachers gave the students little assistance in developing and understanding of what they were doing” (Kilpatrick et.al., 2001, p. 377).

Since Schulman (1986) there are theories of *mathematical knowledge for teaching* and *mathematical quality of instruction*. However, like Ronda and Adler (2019) we do not find empirical answers to important questions: “It is a common view that teacher’s knowledge of mathematics relates to the mathematics made available to learn in instruction. However, there are only a few studies that provide empirical evidence and explanation of how this knowledge is implicated in instruction” (p. 257). Livy and Vale (2011) offer another interesting aspect of this. They start with Ma (1999) who emphasizes that in order to teach mathematics in a successful way, a teacher needs *a profound understanding of fundamental mathematics*. However, that is not enough. It also requires a capacity of identifying “mathematical components within a concept that are fundamental for understanding and applying that concept” (Chick et. al, 2006, p. 299). This means that teachers’ mathematical content knowledge plays a crucial role in in students' learning of mathematics (Ball et.al., 2004; Livy & Vale, 2011).

2. The Present study

The purpose of the study is to analyse effects of teaching and teachers’ mathematical content knowledge about fraction, ratio and proportion, on students’ learning, from grade 4 and 5 to grade 8 in progression.

3. Theoretical approach and Design

3.1. Mathematical content knowledge and teaching

The concept of *Mathematical Content Knowledge* has by Ball, Thames and Phelps (2008), been split up into two main components *Subject Matter Knowledge* (SMK) and *Pedagogical Content Knowledge* (PCK). Moreover, SMK includes three different dimensions: *Common content knowledge* (CCK), *Specialised content knowledge* (SCK) and *Knowledge at the mathematical Horizon* (HCK) (p. 403). This means, that teachers need to understand mathematical concepts, as well as mathematical methods and connections between different part of school mathematics, to partition a content and adapt it to a student’s pre-knowledge and ability. This is in turn a condition for keeping a focus on “the object of learning” and to offer a suitable variation of the content, from a lower to a higher level of difficulty. At the same time, it is important to be aware of, that mathematics taught the during earlier school years, is often based on preliminary, more perceptible, concepts. It is important that such preliminary concepts are gradually developable into correct mathematical concepts. This means that it is not enough for teachers to understand the mathematics they are just teaching. They also need to understand it in such a way, that the content can be unpacked and developed during later school years (Hill et al., 2008). This makes demands on teachers’ ability to overview the progression in students’ learning from grade 1 to grade 9. Against this background, it is important to be aware of, that lack of mathematical content knowledge, like subject matter knowledge, can never be compensated by experiences from practice (Ball & Bass, 2000).

3.2. Variation theory and the object of learning

According to Marton (2015), learning takes place when a student is “capable of being simultaneously and focally aware of other aspects of a phenomenon” (p. 142). In other words, the purpose of teaching is to plan and carry through activities that make it possible for the student to “constitute the object of learning”. For learning to take place, some crucial aspects of the object of learning need to vary, while others need to remain constant. From a teacher’s point of view, this requires a good survey of and insight in the actual content as well as opportunities to identify students’ multiple conceptions of an actual phenomenon. If not, it is neither possible to present a content that enables students to find crucial aspects of the objects of learning, nor to offer them a relevant variation. This means that teachers’ SMK is crucial for their ability to teach and thus for their students’ ability to learn. This is the core of variation theory, and its application of teaching and learning in praxis (Marton & Pang, 2006; Marton, 2015).

Marton (2015) emphasizes that “the object of learning is constituted in the course of learning” (p. 161). To study how an object of learning is understood it is important to relate this to a certain aspect. Inspired by Pong and Morris (2002), researchers have chosen the following aspects: *The expected object* (EO1): What students are expected to learn. *The intended object* (IO): What teachers intend to teach according to the chosen object of learning. *The manifest object* (MO): How the object of learning is related to what really was mediated in the classroom. *The experienced object* (EO2): What the students experienced from teaching. By studying these aspects of the object of learning, it is possible to analyse the quality of interesting aspects of teaching and learning.

3.3. Design of the educational materials

To achieve the purpose of this study an intervention approach was used. The intervention assumed construction of educational materials, EM1 for grades 4 and 5 and EM2 for grade 8 (constructed by researchers), teachers' participation in a seminars-training connected to EM1 and EM 2, and observation of teachers work to implement it. EM1 deals with fractions as equivalent classes and extension of fractions, part-part, part-whole, whole-whole and explanations of how to apply ratio and proportion in problem solving (Suggate, Davis & Gouldning, 2009). EM2 deals with algebraic concepts of ratio and proportionality, and how to use algebraic concepts in problem solving (Ohlsson, 1988). The aim and goal of all tasks for teaching in the EM1 and EM2 are described in detail with focus on crucial aspects of the object of learning, variation of concepts and how to use them in problem solving.

4. Methods

4.1. Participants and data collection

The study included four classes and four teachers who were rated as highly successful by their principals: T1 (21 students) and T2 (20 students) from grade 4, T3 (19 students) from grade 5 and T4 (26 students) from grade 8. The classes were chosen from four schools situated in different suburbs of Stockholm. The reason for this choice of grades was, that rational numbers and proportion are formally introduced in grades 4 or 5 and that algebraic concepts of ratio and proportionality are more formally handled in grade 8. The study was implemented during five or six lessons. Two weeks before that, the teachers were invited to a seminar, were aims and goals were introduced, and the teachers got access to their EM1 or EM2. One week before the study was carried out in a class, the students got a pre-test. During the study all communication between teachers and students were recorded by video and with an extra microphone on the teacher. One week after the study the students got a post-test and a sample of them were interviewed. Collected data was transcribed, categorized, thematized and analysed according to a methodological design and a theoretical approach. The results were structured and categorized according to our four aspects of the object of learning (Pong and Morris, 2002).

5. Results

5.1. Observations of Teaching in grades 4 and 5

The EM1 for grades 4 and 5 contained tasks about extending fractions and fractions as equivalence classes. As an example, the students solved tasks about part-whole and extending of fraction. In this case the crucial aspect was, that when the whole is doubled (tripled), the part will also be doubled (tripled) like in $\frac{2}{3} = \frac{2\cdot 2}{2\cdot 3} = \frac{3\cdot 2}{3\cdot 3}$. To abstract (verbalise) this, students were expected to continuously explain and discuss this process. Teacher T1 was able to structure and carry out such a teaching with focus on the object of learning (IO). The teacher spent almost an hour to ensure that her students understood the basic concepts and after that the teaching went on without problems (EO2). In class T2 and T3, teachers had difficulties in perceiving the object of teaching (IO). In their teaching every single task was treated isolated from its concept and context. Obviously, teachers T2 and T3 were neither able to find the object of learning nor to offer their students any appropriate variation. The result of this was that instead of discussing the idea behind part-whole, extending of fraction or equivalence classes, students were encouraged to guess and after that filling in a number in the numerator when the denominator was doubled or tripled (EO2).

The quality of the communication between teacher and students (MO) was different in the three classrooms. Teacher T1 had focus on the content and guided her class gently towards the chosen objectives. The students were continuously invited to reason about how and why they obtained their answers (EO2). The communication in T2's and T3's classes was mainly student centred at the expense of content. They often questioned "How did you think?", however without any connection to the actual concept and context. In their communication it was more important that every student had a chance to give an answer than to give them adequate feedback to their answers. Moreover, many students were often answering at the same time and were often just guessing (EO2). One consequence of this was that students often took over the initiative and guided the communication in a wrong direction. Another consequence was that there were insufficient possibilities for a real discussion or reasoning (EO2).

5.2. Observations of Teaching in grade 8

The EM2 for grade 8 included tasks dealing with ratio in problem solving like "During a sale the prices were reduced with 10%. Bob paid 63 euro for a pair of shoes. What was the price before the sale?". The aim was to introduce and discuss the concept of proportionality in problem solving and different possibilities to carry out the calculation with different methods (MO). However, teacher T4 directly established "that the best method was" to use the formula $x = \frac{630}{0,90}$ without any explanation of this method/formula and with no focus on the objects of learning or the concept of proportionality (MO). Moreover, there was no possibilities for students to reason about different methods during the lessons (EO2), and there was no variation at all (IO). The main teaching strategy T4 was in fact to discuss just one safe method, cross-multiplication, without explaining when, how and why to use it (MO).

An analysis of the communication in T4's class shows (MO), that the teaching had no focus on the concept of ratio or proportionality. Moreover, most of the teaching was of a kind called "piloting". For an example, teacher T4 wrote $\frac{2}{3} = \frac{x}{12}$ on the board and then asked: "How much is 2 times 12" followed by cross-multiplication: 2 · 12 = 3x. There was no space for students of drawing any conclusions themselves or to reason about different methods (EO2), because teaching T4 did not invite the students to such activities, just piloted them past the crucial aspects of the object of learning.

5.3. Students' knowledge about fractions and proportion

5.3.1. Tests in grades 4 and 5 and 8. The pre-test in grades 4 and 5 showed that some of the students already had some knowledge about fractions and proportion, while most of them had more unstructured and confused perceptions. The post-tests showed that T1's students solved most of the tasks correctly (EO2), while most of T2's and T3's students failed (EO2). A noteworthy observation, which got confirmed during the interviews is, that most students in class T2 and T3 really tried to solve the tasks, but did not understand the concept or context (EO2).

The pre-test in grade 8 showed, that most of the students were able to solve simple tasks on proportion and proportionality but had difficulties with problem solving that presume knowledge about fractions, ratio, proportion and algebraic concepts. The post-tests showed that students had a total focus on calculation, not on significance or concept. For example 11 av 26 students were not able to solve tasks like $\frac{2}{5} = \frac{6}{x}$ or solve a task like "A flagpole gives a shadow that is 6 meters long. Moa who is 1,50 meter tall gives a shadow that is 1 meter long. How tall is the flagpole?". However, the most remarkable was, that none of the students made an outline of the situation with the flagpole. This result confirms what we observed during the observations and the interviews of students, a total focus on calculation, not on concepts or context (EO2).

Another interesting observation was, that many students in grade 8 had the same perceptions of concepts and the same types of misconceptions about fractions and proportion, that were found among the students in classes T2 and T3 in grades 4 and 5. Our interpretation of this is, that teachers T2, T3 were unable to perceive the students' misconceptions. When teacher T4 met such students from middle School, T4 was not able to apprehend their problems. Consequently, the teacher tried to solve the situation by teaching procedural formulas fitting for solving predictable problems.

6. Discussion

The purpose of this study was to analyse the effects of teaching in relations to teachers' mathematical content knowledge of fractions and proportion, and to the progression in students learning from grade 4 and 5 to grade 8. The study showed an obvious connection between teachers' mathematical content knowledge and students' learning and progression in learning (Ball et al., 2008). Most students showed an ability to learn, but their performance was intimately linked to teachers' perception of crucial aspects in teaching, and its variation. In most classes, teachers' mathematical content knowledge was not sufficient to implement EM1 or EM2, despite all preparation (Ball & Bass, 2000). Moreover, in most classes, the teachers offered a very limited space for students in reasoning and identifying the objects of learning and their crucial aspects (Marton, 2015; Pong and Morris, 2002). In addition, anomalies in students' perceptions of basic concepts caused obstacles in their learning. Moreover, some anomalies seem to follow students from middle school to grade 8 (Hill et al., 2008).

7. Conclusions

To sum up, findings from this study show that teachers' mathematical content knowledge and their ability to identify object of learning and apply it in the teaching with focus on mathematical concepts is of vital importance for students learning and progression in learning. Finally, the study also shows how anomalies in students' conceptions may arise if teachers are not aware of earlier misconceptions. The outcome of this study can explain more about crucial challenges in teaching and learning of fraction, ratio and proportion and give support in developing teaching and learning in this area.

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