TEACHING AND LEARNING THE MULTIPLICATION TABLE BY USING MULTIPLICATIVE STRUCTURES: VARIATION AND CRUCIAL PATTERNS

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Abstract

This paper examines and analyzes how students learn multiplication tables, specifically the role of multiplicative structures and how these are used as students learn to master the tables. The analysis is performed in the context of the generalization process related to the teaching activity focusing students' perception of concepts. The theoretical approach applies Davydov's concept of theoretical generalization as perception-conception-elementary concept (PCE model) and Vergnaud's theory of multiplicative structures in three classes: mapping rule (MR), multiplicative comparison (MC), and Cartesian product (CP). For the methodological design, Marton's variation theory has been chosen.
This study includes two teachers and 40 students in two Year 3 classes, followed two years later by one teacher and 25 students in one Year 5 class. The analysis of the outcome is based on documented classroom observations, one-on-one interviews with students and teachers' reflections on students' learning outcomes. The conclusion of the study is that the generalization of multiplication is a difficult process for students, especially in the classes MC and PC, and one that sometimes results in challenges to identifying multiplicative situations and relating these to the multiplication tables. This illustrates that teaching activities and teachers' support are necessary conditions for students' learning. The study also shows that multiplicative structures can help students to find and systematize crucial patterns in the multiplication table, allowing them to learn the multiplication table in a more efficient and structured manner. During the one-on-one interviews, students actively searched for and found structures and solutions that did not come up during lessons. This shows that multiplicative structures are a suitable didactic tool for identifying patterns in multiplication tables, thereby facilitating learning other than by rote.

Keywords: *Multiplicative structures, multiplication table, generalization, student's perception, elementary basic concepts.*

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1. Introduction

Multiplicative structures are one basis for understanding the concept of multiplication and for finding interesting patterns in the multiplication tables. Multiplicative thinking is one of the "big ideas" of mathematics that provide students with tools for learning different content during their early school years. According to Hurst & Hurrell (2014), in primary and middle school the nature of students' learning of multiplication is mostly procedural and their multiplicative thinking and understanding of multiplication may differ. The issue of learning multiplication and multiplication tables through empirical learning has also been addressed by researchers such as Gierdien (2009) and Downton (2015). Their empirical studies show that multiplication teaching is culturally based, often with the emphasis on repeated addition (Askew, 2018; Van Dooren et al., 2010). Earlier studies suggest that students in Years 5, 7 and 9 often intuitively use the primitive model of repeated addition (Fischbein et.al., 1985). Researchers agree that the structural characteristics of multiplication play an important role in learning to understand the concept of multiplication (Park & Nunes, 2001; Sherin & Fuson, 2005). Multiplicative thinking is a complex process to grasp, not only for students but also for teachers. Heng & Sudarskan (2013) have challenged teachers' pedagogical assumptions about what it means to teach for student understanding, with a focus on multiplicative thinking. This implies that, when it comes to multiplication, the organization of teaching must include elements to activate students' learning and to create the conditions for structured learning. The guiding principle of teaching plays a central role and will help students to generalize essential-intuitive-primitive models for multiplication into general, more abstract models. The teaching activity guides students through the various steps of transforming the problem situation and identifying crucial relationships within it. This constitutes the concept of multiplication as a basic level for multiplicative thinking (Kaput, 1985). The development of instructions for promoting mathematical strategies, supporting teachers in understanding mathematical concepts, and helping them to understand when and how students are ready to learn, are a necessary condition for teaching (Ball, Thames & Phelps, 2008). This also implies that students need active support to change and develop their perceptions of multiplication (Vergnaud, 1988).

The development of conceptual thinking and its significance for students' multiplicative thinking is described by Wright (2011), who points out that the student's previous experience of applying the concept is crucial to identifying relationships in different contextual situations, thus activating the student's knowledge as a resource for learning multiplication.

One theoretical approach to conceptualization and generalization is described by FeldmanHall (2018), who emphasizes that generalization is a logical device usually connected with the process of learning. As a teaching method, generalization is closely linked to the process of formatting mathematical concepts as a basis for learning as a mental activity in the transition from perception to concept, e.g., "... a generalization is made – that is, similar qualities in all objects of the same type or class are acknowledged to be general" (Danilov & Esipov, 1957 p. 77). Empirical studies about generalization and conceptualization in the teaching and learning process are described by (Kennedy, 1997; Onwuegbuzie & Leech, 2009; Williams & Young, 2021).

The present empirical study is not investigating an epistemological context related to the generalization of multiplication tables by multiplicative structures as different multiplication models. This study examines the role played by the generalization of multiplicative structures in students learning of the multiplication table, within the framework of Davydov's (1990; 1992) and Vergnaud's (1983) theoretical approach.

2. Theoretical approach and design

Davydov's model emphasizes the connection between generalization and conceptualization. According to Davydov, generalization – a phenomenon related to the mental process – is used to describe different aspects of students' learning. The empirical-theoretical approach taken by Davydov (1990; 1992) indicates that generalization and conceptualization are key components of how schools teach mathematics. As a result of generalization, the student's "...ability to abstract himself from certain particular and varying attributes of an object is noted" (p. 5). This implies that the student's understanding of the nature of facts can be expressed in a verbal response and recognized in a familiar situation. A necessary precondition for the development of the student's ability to generalize and conceptualize is mathematical activities in teaching. These activities must be planned with the emphasis on the content of concept and teacher-student discourse, where students analyze, constitute, recognize and produce verbal responses. Davydov interprets the generalization process as consisting of three linked elements: perception, conception, and concept. The student's apprehension of concept is a result of generalized perceptions and conceptions of many similar objects with a focus on the object's crucial properties. The transition from perception to concept is not an easy process for students of primary school age to grasp. For them, generalization is a form of representation and "elementary concepts". For the purposes of this study, the model of generalization used is "perception–conception-elementary concept" (PCE). According to Davydov (1990), the transition to concept requires that "students master the entire aggregate knowledge about the objects to which the given concept pertains" (p. 12). To make a connection to the theoretical framework about concept as multiplicative structure, it is linked to Vergnaud (1983).

Vergnaud (1983) defines the multiplicative structure of multiplication thus: "Multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects". According to Vergnaud (1983), different multiplicative problems can be described by different multiplication models. These models can be divided into three classes as a base for multiplicative structures: the mapping rule (MR), also known as repeated addition; multiplicative comparison (MC), also known as enlargement; and Cartesian product (CP) including properties of commutativity. These models are used as a context for this study.

3. The study

The study is a part of a project focused on how students learn multiplication tables using multiplicative structures (MR, MC and CP), as well as the role of multiplicative structures for Year 3 and 5 students learning the multiplication table using PCE theory.

4. Methods

4.1. Variation theory

Variation theory (Marton, 2015) is a general theory of learning with its roots in phenomenography. For learning to take place, there must be a focus on the crucial aspects of the object of learning. At the same time, there must be some variation in some crucial aspects while others remain constant. From the teachers' point of view, this requires a good overview of and insight into the content, as well as knowledge of the subject in question (Shulman, 1986), otherwise the teacher will be neither able to plan sustainable teaching nor identify the crucial aspects of students' conceptions of the actual phenomenon. In this study, variation theory is used as a methodological design to analyze the variation of teaching activities with the emphasis on multiplicative structures and crucial patterns in the multiplication table related to multiplicative structures.

4.2. Data collection

Participants in the study were two teachers and 40 students in two Year 3 classes, followed two years later by 25 of the same students, now in Year 5. Data were collected by observing the teaching process and conducting one-on-one interviews with students in Year 3 and then later in Year 5. All data were transcribed and systematized for analysis related to the theoretical tools. In Year 5, students were asked the same questions about multiplication and multiplicative problems as in Year 3.

5. Findings from the study

5.1. Finding 1

During one lesson 1 in Year 3, the teaching activity was based on Figure 1. The task was formulated thus: *Describe the picture as addition and multiplication.*

Figure 1. Picture from the textbook Favorite Mathematics (2013).

A sample of the student's answers is:

Student 1 *3 + 3 + 3 + 3 = 12*
Student 2 *4 × 3 = 12*
Student 3 *3 × 4 = 12*
Student 4 *3 × 12 = 12*
Student 5 *2 × 6 = 12*

The answers demonstrate the students' varying perceptions. While the focus was on different ways of explaining multiplicative situations through the three multiplicative structures, the teacher-student discourse was a communication in triads, question-answer-reaction, and offered limited possibilities for the teacher to ascertain in whether the answers achieved consensus. For example, the answer 2 × 6 = 12 was rejected although there are 6 birds on each side of the trunk, and the teacher did not pay attention to the connection between 4 × 3 = 12 and 3 × 4 = 12, the commutative law for multiplication. It is in fact difficult to find any clear goal for that part of the lesson.

5.2. Finding 2

The results of observations and interviews suggest that most of the students perceived MR as a multiplicative structure. That said, it was difficult for most of them, even in Year 5, to connect this to an MC structure and use it to analyze the structure of the multiplication table, CP. One reason for this is the problem in perceiving the commutative law for multiplication in a repeated addition.

5.3. Finding 3

Few of the students in Year 5 were able to perceive the property of commutativity. During the interviews, one question was: Which sum is larger, 4 + 4 + 4 + 4 + 4 + 4 + 4 or 7 + 7 + 7 + 7? Only four of the students were able to understand these repeated additions as multiplications and use the commutative law for multiplication. The other students reasoned as follows:

Student 1 *It* is *7 + 7 + 7 + 7*
Teacher *Why, can you explain?*
Student 1 *Because 7 is a bigger number than 4*
Student 2 *I think it is the one with 4's*
Teacher *Can you explain why?*
Student 2 *4 + 4 equals 8. And there are just four 7s. So, my feeling is that this one is bigger*

An understanding of commutativity is a crucial aspect of the multiplication table. This shows the importance of developing MR into MC and CP to understand a multiplicative property like commutativity and later associativity and distributivity.

5.4. Finding 4 Multiplicative structures and crucial patterns in the multiplication table

Interviews with Year 5 students demonstrate that most of the students were able to read the multiplication table using MR and MC structures. However, at the beginning of the interviews few students were able to identify CP structures and use them to find crucial patterns in the multiplication table; not even the four students who knew the multiplication table by heart. During the interviews, the students were made aware of one or two CP structures, after which most of them were able to find relationships between even and odd products, commutativity as symmetry in the table, and sometimes even patterns in the five- and nine-times tables. While students clearly had no problem understanding multiplicative structures in the multiplication tables, unless these are highlighted, they will not notice them.

6. Discussion and conclusions

The study shows that the generalization of multiplication, especially from MR to MC and CP, is a difficult process for students to grasp. This illustrates the fact that teaching activities and support from teachers are necessary preconditions for students' learning. The study also shows that multiplicative structures, such as even and odd functions and commutativity, can help students to find and systematize crucial patterns in the multiplication table and thus learn the multiplication table in a more efficient and structured manner. In interviews, students actively searched for and identified structures and solutions to problems that had not come up during lessons. Multiplicative structures are suitable didactical tools to support students in finding patterns in the multiplication table, thereby facilitating their learning of basic facts, not only by rote but also in an algebraic context.

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