ARITHMETIC AND ALGEBRAIC KNOWLEDGE IN STUDENT LEARNING OF CONCEPTS

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Abstract

Current research deals with students' arithmetical and algebraical knowledge, with a focus on a conceptual connection, and the relationship between two aspects of knowledge. The content in question is rational numbers, rational equations, and problem-solving in terms of proportion and ratio in grades 7, 8 and 9. The method contains three tests given to 400 students in grades 7–9. Tools for analysis were theories of generalizing arithmetic into algebra (Kieran, 2004), and the relationship between arithmetic and algebra in a conceptual context (Kaput, 2008).
Current research shows that student knowledge of algebra and arithmetic often has a limited conceptual connection, and a weak relationship. Their knowledge of arithmetic operations and solving rational equations used to be just procedural, and reliant on formulas learnt in a procedural way, and often mixed up. The study also shows that student procedural strategies for finding formulas suitable for solving the equations, as well as carrying out the corresponding calculations, were often insufficient.
The study investigates shortcomings in students' conceptualization of arithmetic operations with rational numbers, and how to apply them to solving rational equations. One reason for this, is lack of continuity in instruction and learning.

Keywords: *Rational numbers and algebra, conceptual knowledge, students' arithmetic and algebraic knowledge.*

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1. Introduction

An important element of teacher training is that student teachers develop algebraic reasoning ability, based on generalizing mathematical ideas, and linked to algebraic concepts (Blanton & Kaput, 2005). This applies not least to the concepts constituting the basis of modern algebra and conceptual relationships between algebra, and the generalization of arithmetic, algebra and patterns, algebra and mathematical models, as well as the meaning of algebraic symbols (Kaput, 2008).

The generalization of algebraic concepts and the ability to create meaning from symbols is a long-term process, linked to the expansion of student arithmetic knowledge (Kieran, 2007). Algebraic reasoning is also important for conceptualizing algebra and using it to expand mathematical knowledge into abstract algebraic knowledge. According to Mason (2008), generalization of algebraic patterns calls for conceptual based knowledge, and the ability to analyze arithmetic situations. This means that student learning of algebra, related to earlier experience of learning and conceptual knowledge, plays a crucial role in operating with rational numbers and solving rational equations (Hackenberg & Lee, 2015). A prerequisite for this is that teachers can take a perspective on student learning, such that the continuity in, and expansion of, algebra in student learning includes conceptual relationships between different number areas from natural numbers to real numbers. This means, among other things, that the students understand conceptual relationships previously used for natural numbers, in a way that can be generalized to negative, rational and real numbers, even if the operations themselves must be modified. At the same time, it is important that students perceive subtraction as the inverse operation of addition, and division as the inverse operation of multiplication. To help students make such generalizations, the teacher is required to have sufficient knowledge of algebra, to understand how an extension of arithmetic works in a conceptual sense, before they start teaching that content (Kieran, 2004). It also involves how students can learn algebra by working informally with the four rules of arithmetic methods in grades 1-3 and 4-6, but in such a way that they will later be able to apply this to whole numbers, rational numbers and real numbers. To understand these generalization processes, students need pre-knowledge about characteristics of rational numbers before they apply rational numbers in problem solving.

2. Literature view

2.1. Student pre-knowledge

Arithmetics taught during early school years is often based on preliminary, more perceptible concepts, and it is important that these preliminary arithmetic concepts are gradually developable into correct mathematical concepts. This is often carried out with metaphors or by using different representations, for example, pictures. However, according to Kinard and Kozulin (2008), the aim of all representations is abstraction, student verbal understanding of arithmetical concepts and their crucial properties.

Learning of rational numbers is a matter of conceptual meaning (Ni & Zhou, 2005; Gözde & Dilek, 2017) from multiplication to fractions and more, a process that successively presupposes adequate prerequisite knowledge of algebra. According to Vygotsky (1986), mathematics is a social construct that implies an ability for abstract thinking. For that reason, students are not able to learn mathematics without support from their teacher.

Student understanding of rational numbers as arithmetical concepts assumes an ability to think in terms of algebraic abstracts. For students to assimilate the abstract concept of fractions, there is often a need for some kind of representation, a variation of tasks and problem-solving. Its aim is to facilitate a verbalization of crucial properties. However, as Ohlsson (1988) emphasizes, fraction is often a "bewildering array", and it is important for a student to know which property of rational numbers is currently represented. For this reason, it is important for students to have suitable prerequisite arithmetic knowledge (Zazkis & Liljedahl, 2002; Kieran, 2018; Kieran & Martínez-Hernández, 2022). Moreover, when students are introduced to a new phenomenon, they are often more inclined to assimilate it according to their current understanding, than to accommodate and develop a new, deeper understanding.

2.2. Conceptual continuity in instruction and learning

Mathematics is abstract and has left the specific origin of the problems. This, in turn, is a condition for being general i.e., applicable in a variety of situations. An important follow-up question is what is meant by abstract and abstraction. Skemp (1987) explains the meaning of the terms, linked to school mathematics as follows:

> Abstracting is an activity by which we become aware of similarities among our experiences. Abstraction is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having similarities of already formed classes. (p. 21).

That mathematics is abstract and general does not only apply to the academic subject of mathematics, but also to school mathematics. 2+1=3 is an abstraction that is general in the sense that it is applicable no matter what objects you add, and not only objects, but also minutes, ideas, age, etc. It is important to pay attention to this in student learning, as well in formal studies in mathematics, and continuous reflection on relationships between arithmetic and algebra, as how the complex nature of the content within arithmetic can be expressed as algebra. This will create conditions for continuity in student learning, and provide the pre-knowledge needed for understanding algebra (Carraher, Schliemann, Brizuela, & Earnest, 2006).

A central aspect of mathematics is the field of algebra. The common student perception of algebra is that it is about complicated "counting with letters". In fact, basic algebra deals with the conditions for the arithmetic operations that students are already learning informally in the first years of school, and how later they can use it to derive and operate with negative numbers and numbers in fractional form. The "letters" are only used to describe the fact that something is general. To describe what is meant by an equation of the first degree does not require all such equations to be written down. Using symbols, this can be written as $ax + b = 0$, where $a \neq 0$. The conceptual relationship between rational numbers and equations is important in student learning of algebraic symbols and abstracting the ideas behind them (Kieran, 2013; Karlsson & Kilborn, 2014).

3. Theoretical framework

3.1. Generalizing arithmetic into algebra

An important feature of teacher training is that student teachers develop skills in algebraic reasoning based on generalizing mathematical ideas, linked to algebraic concepts (Blanton & Kaput, 2005). This applies not least, to concepts that constitute the basis of modern algebra, and conceptual relationships between algebra and the generalization of arithmetic, algebra and patterns, algebra and mathematical models and the meaning of algebraic symbols (Kaput, 2008). For students to understand symbols and abstract algebra, they need to generalize algebraic concepts by reasoning with symbols (Kaput, 2008). Students' ability to express themselves using algebra and to make transformations from arithmetic concepts into algebraic concepts, depends on their conceptual knowledge of the relationship between arithmetic and algebraic concepts, and how numbers are transformed into algebraic symbols. For instance, students' conceptual knowledge of rational numbers is key to understanding equations, their construction and their conceptual meaning.

According to Kieran (2004) generalization of algebra requires algebraic activities with a focus on the ability of the student to explain and express their knowledge and understanding. Such activities include several main components: (1) generalization of arithmetic concept; (2) conceptual transformation from arithmetic into algebra; and (3) analyzing and applying in problem solving. Mastering algebra means not only knowing different algebraic expressions and equations, but also understanding conceptual connections between numbers and expressions, and between numbers and equations, as tools in problem solving. This means that mastering algebra not only includes a path from separate algebraic expressions and equations to their generalizations, but also the way back - from generalization to arithmetic.

The transformation of student knowledge from arithmetic to algebra presupposes a fundamental understanding of crucial properties and representations of numbers, and their connection to algebraic expressions and equations. Important in Kieran's view of this is that student mastery of algebra knowledge includes an ability to apply their conceptual knowledge to different problem-solving situations. Such a systematic pattern in student learning can effectively help them understand the conceptual relationship between arithmetic and algebra, and how to use it in problem solving. For example, students' conceptual understanding of rational numbers as equivalence classes. Such as $\frac{1}{2} = \frac{2}{4} = \frac{2}{4}$ etc., is a conceptual pre-knowledge in understanding an operation such as extension of rational numbers, conceptual understanding of symbols, and the conceptual meaning of equations like $\frac{x}{2} = \frac{2}{4}$. This kind of knowledge also means that a student can comprehend and solve such equations using algebraic reasoning, and without using formulas (Carpenter, & Levi, 2000; Karlsson & Kilborn, 2015).

4. The purpose

The purpose of the study was to examine student arithmetic and algebraic knowledge of rational numbers, and their ability to use this in problem solving. The research questions are: (RQ1) How do students interpret and represent rational numbers? (RQ2) How do students handle transitions from rational numbers to symbols and rational equations? and (RQ3) How do students apply this in a problem-solving situation?

5. Methods

5.1. Participants and procedure

The study was design to examine student arithmetic and algebraic knowledge in a conceptual context with especial focus on student perception of rational numbers and their properties, and how to handle this in solving rational equations and problems dealing with proportion and rate. Participants were 400 students in grades 7, 8 and 9, with three teachers A, B and C. In grade 7 two classes participated with teachers A and B, and one class with teacher C. In grade 8 one class participated with teacher A, two classes with teachers B and C. In grade 9 two classes participated with teachers A and C, and one class with teacher B.

The study included a quantitative and a qualitative approach. The instrument consists of three diagnostic tests: DT1, DT2 and DT3. Test DT1 focused on representations of rational numbers and operations with rational numbers, test DT2 focused on algebraic equations like $\frac{3}{5} = \frac{x}{8}$, and test DT3 focused on problem solving related to proportion and ratio. Each test consists of 7 tasks of increasing complexity. The tests were designed with two empty spaces, one for calculation and the other for written explanations. The quantitative approach concerned the frequency of correct answers and the qualitative approach concerned the quality of student answers, as well as a conceptions and misconceptions.

6. Data analysis

The main purpose of the study was to answer research questions RQ1, RQ2 and RQ3 about student conceptual understanding of rational number and rational equations, and their ability to use this in problem solving. The theoretical model was based on Kaput (2008) and Kieran (2004), and was used to report the results of the diagnostic tests in a conceptual meaning. Moreover, the results were interpreted and explained in terms of written recommendation intended to develop the skills of the teachers involved and their colleagues.

7. Results and discussion

Conceptualization of arithmetic as rational numbers in student learning and its transformation into algebra has been recognized as a crucial and difficult issue in student mathematical learning. This study illustrates that students' arithmetic pre-knowledge (Zazkis & Liljedahl, 2002; Kieran & Martínez-Hernández, 2022), and their pre-knowledge of rational numbers (Ni & Zhou, 2005; Gözde & Dilek, 2017) play an important role for in student achievement of algebraic equations and problem solving, and more generally in student learning of the abstract nature of algebra, expressed in symbols (Carraher et al., 2006).

The test data shows a low development from grade 7 to 9 of student ability to handle fractions, rational equations, and algebraic reasoning. In grade 9, almost all students relied on formulas, for example, to solve simple tasks such as $2 \cdot \frac{3}{7}$. Moreover, 40% of the students in grade 9 failed. The low ability in terms of algebraic reasoning also became clear in problem solving. Most students just tried to apply a formula that they did not know how to handle. One example is: "*Anna can cycle 80 kilometers in 3 hours. How long does it take Anna to cycle 50 kilometers at the same speed*?" Only 26% of the students in grade 9 were able to solve it. Their solutions and comments on the tests show that most of the students were unable to reason, to choose a correct formula, or to carry out the calculation. When comparing the solutions from grade 7 to those of grade 9, it became obvious that there had been very little development of knowledge from grade 7 to 9. In grade 7, the students already used formulas like in grade 9, but most of them were unable to handle them. The problem is that such procedural knowledge is an insufficient ground for developing algebraic reasoning (Kieran, 2004).

One question in test DT2 was "*For what values of x and y are* $\frac{5}{6} = \frac{x}{y}$". The response rate was 2% in grade 7, 7% in grade 8, and 3% in grade 9, who answered x = 10, and y = 12. This confirms a lack of both reasoning ability and conceptual understanding of fractions (Kaput, 2008). It also shows student problems in understanding the important property of fractions as equivalence classes, a gateway to understanding and solving the current rational equations.

8. Conclusions

The study shows student conceptual knowledge of rational numbers, how they handled rational equations, and use of fractions and equations in problem-solving. One outcome is that the transition from arithmetic to algebra is a difficult process and impossible to carry through with only procedural knowledge. The theoretical frameworks of Kieran's (2004) and Kaput's (2008) visualize fundamental limitations in student solutions for equation solution, and their dependence on conceptual knowledge. More specifically, generalization of algebra cannot take place without generalization of arithmetical concepts (rational numbers). However, conceptual knowledge of rational numbers implies student ability to carry through solutions for equations through reflection and reasoning, even without the use of formulas.

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