PRE-SERVICE TEACHERS' KNOWLEDGE OF MATHEMATICS: A FRAMEWORK FOR SUSTAINABLE DEVELOPMENT OF STUDENT KNOWLEDGE

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Abstract

The purpose of this article is to draw attention to, analyze and discuss the following issues: (1) What mathematics should teacher education include, in order for student teachers to gain knowledge of a teaching practice that ensures the progression in students' mathematics development, and (2) How can the subject-specific content in an algebra course for student teachers be designed through an interaction between formal concepts in mathematics and the content of practical mathematics teaching with focus on algebra. An analysis of these issues is carried out within a theoretical framework of didactics of mathematics, related to a research context. This article is based on two research projects, MIL (Mathematics in teacher education) and SKUM (Student teachers' knowledge and perceptions of mathematics) as well as ongoing research work with a focus on the quality of student teacher education in mathematics and the didactics of mathematics in the K–3 and 4–6 programs at Södertörn University.

Keywords: *SMK model and pre-service teacher knowledge in mathematics, student teacher learning for teaching, algebra in teacher education, Abelian groups and teaching, rational numbers and teaching.*

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1. Introduction

Mathematics is part of a cultural heritage and is an important tool for perceiving and developing the increasingly complicated world around us. Therefore, a primary purpose of teacher education in mathematics is to provide a perspective on the teaching of mathematics that is characterized by both science and how science can be implemented in school in relation to students' learning. This means that teacher education should focus both on the development of teacher students' own knowledge of mathematics and on how this can be translated into teaching in school in terms of how students learn mathematics at different ages (Hill et al., 2008). This applies not least to how they can present mathematics in a well-structured way, based on its concrete origins. The goal is, that the student teachers perceive the importance of what students gradually learn during their first years of school, based on individual students' abilities and needs, which will be generalized in the direction of the academic subject of mathematics (Subramaniam, 2019). This implies, that the mathematics taught during the first years of schooling must be preliminary, and must be, a simplified form of mathematics, that makes it accessible to all students, but at the same time is based on sustainable and developable mathematical concepts and methods. Accessibility also deals with student teachers' ability to find a continuity in students' learning from pre-school to grade 9 and onwards. This means that students successively learn the internal structures of the concepts as essential properties which in turn must be generalized with the aim of understanding the significance of the concepts and their connections and relationships to other concepts. To perceive and follow such a learning process in students' learning requires that the student teachers themselves are able to process and produce knowledge in their own learning. In other words, they must be able to take a second-order perspective on students' learning, which is intimately related to a first-order perspective on their own learning of mathematics (Leatham, 2006). This is a matter of solid self-awareness of how knowledge is perceived and developed, and what misconceptions may arise in learning. Pupils' misconceptions of mathematical concepts and methods can, like incorrect generalizations during earlier school years, cause serious consequences when students reach secondary school. To promote students' learning of mathematics and to perceive and correct their misconceptions, requires that student teachers have solid knowledge of the current mathematics, related to practical teaching.

This complexity also requires the transformation of formal mathematical concepts to a level that enables teacher students to learn. From this point of view, it requires a good overview of and insight into the actual content, and an ability to break down the content, thus taking the teacher students' individual perceptions of the actual concept into account (Askew 2008). In addition, (Ball & Bass, 2000) emphasize that a lack of teacher knowledge about mathematical content, as well as subject matter knowledge in mathematics, can never be compensated for by practical experience. This means that today's teacher education should aim to prepare student teachers in such a way that they are able to teach a form of mathematics that supports the mathematical development of the students. Another aspect is, that the impact of the student teachers´ mathematical knowledge and their experience from their studies in mathematics from primary and secondary school is very strong and crucial for their reflections about the opportunity to study mathematics and the didactics of mathematics in teacher education. The same aspects are problematized also by other researchers (Radovic, Black, Williams, & Salas, 2018).

2. Background

In this article the results of two ongoing theoretical research works are followed up: namely *Student teacher knowledge and perceptions of mathematic* (SKUM) and *Mathematics in Teacher Education* (MIL) (Karlsson, 2015). The purpose of the theoretical study is to analyze what mathematics in teacher education means with respect to student teachers' knowledge of mathematical content, and the sustainable development of students' mathematical knowledge in elementary school, with a particular focus on algebra. This project focuses on the following research questions. *RQ1* What mathematics should teacher education include, in order for student teachers to gain knowledge of a teaching practice that ensures the progression in students' mathematics development. *RQ2* How can the subject-specific content in an algebra course for student teachers be designed through an interaction between formal concepts in mathematics and the content of practical mathematics teaching with focus on algebra.

2.1. The Subject Matter Knowledge model for pre-service teachers' learning of mathematics

Over the years, several researchers have claimed that teachers' knowledge of mathematics and knowledge of teaching is not sufficient to develop students' learning in mathematics. This led to a need to develop a "practice-based" theory called Subject Matter Knowledge (SMK). This theory forms the basis for what mathematics teachers should be able to teach. According to Ball, Thames and Phelps (2008) and Hill et al, (2008), the SMK model is divided into three areas, namely Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK).

> *"Common content knowledge is held by an adult who can use a method to solve a mathematical problem whereas specialized content knowledge is mathematical knowledge that is unique to teaching"* (Ball et al., 2008, p. 399).

Ball et al. (2008) emphasizes that teachers' CCK is a necessary factor for teaching, but it requires an interaction with SCK. One interpretation of this is that CCK is about the student's perspective of the content while SCK is about the teacher's perspective of the same content. SCK is a prerequisite for keeping focus on the "learning object" and offering a suitable variation of the content, from lower to higher levels of difficulty. At the same time, it is important to be aware that mathematics taught in the earlier school years is often based on preliminary concepts that will gradually be developed into more correct mathematical concepts. This means that it is not enough for teachers to simply understand the mathematics they teach. They also need to understand it in such a way that the content can be unpacked and developed in later school years. This put demands on teachers' ability to summarize the progression of students' learning from year 1 and onwards, in order to ensure the progression in teaching, which in turn leads to a need for knowledge in HCK, including knowledge of the curriculum in mathematics. Consequently, HCK is about seeing mathematics in a wider perspective, not least how the mathematics taught to younger children is connected to teaching at later stages and vice versa. It is also about how basic mathematical patterns (structures) permeate mathematics at all stages.

2.2. The SMK model from a Swedish perspective

In the same year that Shulman (1986) described what teachers should be able to teach, a research review in three parts called *Fackdidaktik I – III* was published in Sweden (Marton, 1986). Already a basic view of the SMK was presented. Under the heading "The origin of subject- didactic knowledge", Marton wrote that one can easily get the impression that one always achieves subject-didactic knowledge through the application of general didactic theories being applied to different subject matters. This is by no means the case. The subject-didactic specialties are autonomous in the sense that, although they can borrow theories and models from neighboring sciences, especially from general didactics, both description and theory formation must be developed from within, based on the special content. Marton described the research on subject didactics that was conducted in Gothenburg in the 1970s (Kilborn, 1979; Lybeck, 1981) and stated that studies of what and how the aspects of the teaching that take their point of departure in a mapping of the qualitatively different ways in which students perceive the phenomena covered in teaching, can be considered a research specialization within the didactic field. This direction can tentatively be called phenomenographic didactics (p. 68). A summary of this work is that what many years later was called SMK was a well-established area in Sweden already in the 1980s, but for various reasons, was not developed further as mathematics didactic research but instead was developed in the form of phenomenographic research (Marton, 2015).

3. Algebra in teacher education. Why algebra?

An important feature of teacher training is that student teachers develop algebraic reasoning based on generalize mathematical ideas, linked to algebraic concepts. This applies not least to concepts that constitute the basis of modern algebra and the conceptual relationships between algebra and the generalization of arithmetic, algebra and patterns, algebra and mathematical models and the meaning of algebraic symbols (Kaput, 2008). This presupposes that the student teachers can take a teacher's perspective on students' learning so that the continuity in, and expansion of, algebra in students' learning includes conceptual relationships between different number areas from natural numbers to real numbers. This means, among other things, that the conceptual relationships previously used for natural numbers also apply to negative numbers, rational numbers and real numbers, even if the operations themselves need to be modified. At the same time, it is important to perceive subtraction as the inverse operation of addition, and division as the inverse operation of multiplication. To help students make such generalizations, it is necessary for the student teachers to be provided with sufficient knowledge of algebra in order to understand how an extension of arithmetic in a conceptual sense works, before they start to teach that content (Kieran, 2004). It is about how students can learn algebra by working informally with the four rules of arithmetic methods and natural numbers in grades 1-6, but in such a way that they will later be able to apply these to whole, rational and real numbers. To understand these generalization processes, the student teachers need meta-knowledge of algebra.

3.1. Abelian groups

Basic arithmetic assumes two Abelian groups (van der Waerden, 1971), one for addition and one for multiplication. A group consists of a set, for example natural numbers, and an operation, such as addition. The following conditions apply to addition as *Abelian group for addition*.

- For all a and b in the group, the sum $a + b$ also belongs to the group. The group is said to be closed under addition.
- For all a and b in the group, $a + b = b + a$. This is the commutative law.
- For all a, b and c in the group, $(a + b) + c = a + (b + c)$. This is the associative law.
- There is a neutral element 0 such that $a + 0 = a$ for all a in the group.
- For all a in the group there is an element $(-a)$ in the group such that $a + (-a) = (-a) + a = 0$. Here $(-a)$ is called the (additive) inverse of a and vice versa.

Understanding the meaning of the Abelian groups is a key to algebraic ideas and logic. At the same time, it gives the student teacher an understanding of why students should learn mathematical concepts, what is decisive when introducing mathematical concepts and what is essential characteristics of the concept. A closer analysis of the *Abelian group for addition* shows that the first three points give information about which addition operations can be performed and how an addition algorithm can be built. This provides an important knowledge of what content the teaching ought to include with focus on what students should learn about algebra. Concerning the natural numbers, these three points are early known to all students, at least informally. To get further and understand subtraction and how to work with negative numbers, the last two points become important. For every natural number such as 4, there is an inverse, an opposite number, (-4). In this way, not only the negative numbers are defined, but also subtraction of whole numbers. The subtraction $a – b$ can be defined as $a + (-b)$. Based on this, it is possible to explain why $a – (-b) = a + b$. Simply stating procedurally that "same signs give plus and different signs give minus" does not lead to any developable knowledge. Moreover, the two minus signs have completely different meanings, they just look similar.

There is another *Abelian group for multiplication*.

- For all a and b in the group, the product $a \cdot b$ also belongs to the group. The group is said to be closed under multiplication.
- For all a and b in the group, $a \cdot b = b \cdot a$. This is the commutative law.
- For all a, b and c in the group, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This is the associative law.
- There is a neutral element 1 such that $a \cdot 1 = a$ for all a in the group.

- For all a in the group (provided that $a \neq 0$), there is an element $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$. $\frac{1}{a}$ is called the (multiplicative) inverse of a and vice versa.

In order to link addition to multiplication, there is the distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$. An analysis of the definition for multiplication gives the student teachers an understanding that multiplication is an operation in arithmetic with special properties. The student teachers can also understand how important the distributive law is, not only to explain how the multiplication algorithm is structured but also its important role in mental arithmetic. By studying the definition of the group for multiplication, the student teachers can also realize the risks with a one-sided definition of multiplication as repeated addition. The commutative and associative laws of multiplication are difficult to derive from repeated addition, because of its one-dimensional nature. Repeated addition does not provide any insight into the two-dimensional structure of multiplication. As with addition, students early on master the first three points in the Abelian group for multiplication, when it comes to natural numbers, at least informally, although far from everyone understands the structure and properties of multiplication. The last two points in the definition deal with the inverse (reversed) operation of multiplication. With the help of the inverse, the student teachers can understand an important property of multiplication, namely how, starting from the natural numbers, they can not only define the rule of division of a / b as $a \cdot \frac{1}{b}$, but also the basic fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ At the same time, the meaning of the multiplication operation must be redefined when the set of numbers is extended from whole numbers to rational numbers. To understand this and thereby create continuity in teaching, it is important that student teachers at all stages study basic algebraic concepts. This is a CCK which is a necessary basis for understanding the SMK which in turn provides the basis for a teacher student's ability to teach algebra in school.

3.2. Rational numbers, proportionality, and algebra

According to van der Waerden, (1971),

- Rational numbers are defined by quotients of integers a and b, where $\frac{a}{b} = a \cdot \frac{1}{b} = \frac{1}{b} \cdot a$ and $b \neq 0$. Important properties of the rational numbers are that they form equivalence classes like $\frac{a}{b} = \frac{c}{d}$, where $a \cdot d = b \cdot c$. Each such equivalence class consists of infinite numbers of the type $\frac{2}{5} = \frac{4}{10} = \frac{6}{15} = \frac{8}{20} = \frac{10}{25} = \frac{12}{30} \ldots$
- They form a commutative field of numbers, which means that all the arithmetic laws that apply to hole numbers also apply to rational numbers. This means, that you do not have to start over from square 1 every time you change number range.
- Two new rules for calculation, must be introduced, namely $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

This shows an important property of the rational numbers, namely that they are divided into equivalence classes, which means that every rational number in fractional form can be written in an infinite number of ways. This in turn means that there is a connection between rational numbers as fractions, proportions, ratios and rational equations (Kieren, 1976). An extension of arithmetic to algebra should thus include this. Another essential property is that the same arithmetic laws, the commutative, the associative and the distributive laws, also apply to rational numbers. This means that the arithmetic laws are a necessary key for the extension of number ranges from natural numbers to real (and complex) numbers and an algebra where the laws of arithmetic are expressed with symbols. This algebraic definition of rational numbers illustrates the importance of knowledge of mathematics (CCK) which in turn leads to knowledge of teaching content.

4. Conclusions and discussions

This article discusses findings regarding what mathematics in teacher training can mean in relation to mathematics didactic research such as the SMK model and how an interaction between formal mathematics and school mathematics with a focus on algebra can be achieved. Against this background, the student teachers can be offered a mathematical content that in turn can be transformed into teaching practice and in the long run benefit their future students. This article also draws attention to the need for the development of teacher training, where connections to research can be used when designing courses in mathematics didactics for future mathematics teachers. This, in turn, can be the start of a life-long development of teachers' competences. This article points out that student teachers' own knowledge of mathematics and algebra (CCK) is a key to their understanding of what the content in teaching is about and what is meant by continuity in and sequencing of the content in mathematics teaching. In addition to this, a focus in teacher education should be on student teachers' knowledge of mathematics, where their knowledge of the practice of teaching is a logical continuation of their own knowledge of mathematics in combination with theories and empirics about mathematics teaching. An important conclusion is that an interaction between practice and a theoretical foundation can lead to meaningful learning for the student teachers and, in the long run, for their students' learning in mathematics at school.

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