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TEACHING FRACTIONS AND THE CONCEPT OF INVERSE OPERATIONS: SCIENTIFIC CONCEPTS IN PRE-SERVICE TEACHERS' LEARNING OF MATHEMATICS FOR TEACHING PURPOSES

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Abstract

The purpose of this project is to analyze how the scientific concept of inverse operations can be used as a "bottom-up" approach for teaching mathematical operations with whole and rational numbers. The primary aim of this analytical review is to provide support for student teachers in their learning of scientific mathematical concepts for the purposes of teaching. The theoretical approaches applied in this project are the theories of mathematical structures, especially the theory of inverse semigroups (Abelian groups), as well as relational thinking in comparison with instrumental thinking. The methodological approach is Vygotsky's Doctrine of Scientific Concepts. The presentation of the analytical findings is intended to illustrate a clear connection to mathematical structures, such as the concepts of inverse, as well as how these structures can support pre-service teachers' learning with regard to teaching mathematical operations in arithmetic and algebra. The development of a theoretical approach based on this study and its analytical findings is ongoing, but it is already being implemented in the teacher education programs for Grades K-3 and 4-6 at Södertörn University in Sweden.

Keywords: *Learning for teaching, mathematical structure, scientific concept, rational numbers, inverse operations.*

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1. Introduction

International research in this field is constantly developing. Researchers agree that the quality of mathematics education received by prospective teachers plays a central role in their subsequent students' ability to acquire math skills. The findings of international empirical and theoretical studies imply that teachers' knowledge of mathematics and the way they teach it do not provide sufficient support for their students' development in the subject (Ball, Hill & Bass, 2005).

Numerous researchers point out that the pre-service preparation of math teachers should involve interning at a school, so they can gain insights into how students learn math and mathematical context. At the same time, pre-service teachers' learning of math and the didactics of mathematics is crucial for their ability to teach, because teachers' mathematical knowledge corresponds to students' development and performance (Hill, Ball, & Schilling, 2008; Mallart, Font, & Diez, 2017). What do pre-service teachers learn about mathematics, and how should they learn mathematical concepts that provide understanding of their own learning and that of their students? How should the content of pre-service teachers' mathematics coursework be structured to ensure that student teachers understand core mathematical concepts and are able to see and use patterns and relationships within these concepts as a basis for breaking down their meaning into essential parts via a process of analysis and integration? Many studies on pre-service mathematics learning have addressed these questions and identified a need for more knowledge about the relationships between fundamental mathematics instruction through mathematical concepts and the content of teaching material for student teachers that gives them the skills to teach math via necessary progression and logic.

According to Vygotsky (1986): "The only good kind of instruction is that which marches ahead of development and leads it" (p. 188). A key to pre-service teachers' acquisition of mathematical knowledge for teaching purposes is that they must learn fundamental mathematics and didactics in well-organized courses with clear, scientifically based contextualization that ensures the development of solid math knowledge specifically geared toward teaching. Researchers agree that teaching basic arithmetic is by no means easy and straightforward; it demands theoretical knowledge about the content.

According to Kieran (2004), structure sense in arithmetic is a conceptual bridge to algebra and other areas of mathematics. Knowledge about numbers and mathematical operations is a key to ensuring the continuity of learning, the development of prerequisites for mathematical thinking, and the ability to re-construct mathematics (Kozulin, Gindis, Ageyev, and Miller, 2003). From this point of view, a teacher or pre-service teacher needs a good overview of (and good insight into) the current content of operations, as well as opportunities to identify multifaceted perceptions of the operations and continuity of thinking. At the same time, it is important to give them opportunities to understand the nature of school mathematics in relation to mathematics as a science. According to Kozulin et al. (2003) conceptual change in learning is the result of students' perception of scientific (mathematical) concepts and ability to ascribe language to their thoughts. This is a central point of view in Vygotsky's approach (1986) to theoretical learning.

The purpose of this theoretical project is to analyze how the mathematical concept of inverse operations tied to rational numbers can be used as a bottom-up approach in how pre-service teachers learn math for the purposes of teaching. The aim is to support student teachers' development of knowledge about mathematical structures in terms of their role as scientific mathematical concepts for teaching.

2. Theoretical perspectives

2.1. Mathematical structure and mathematical thinking

Mathematical structures are crucial to pre-service teachers' knowledge of mathematics and ability to teach the subject. In this context, mathematical structures spur deeper mathematical thinking and help learners to engage in the learning process more effectively (Mason, Stephens, & Watson, 2009). In terms of how we think, the analysis of mathematical structures can also elucidate the connection between "conceptual and procedural approaches to teaching and learning mathematics" (p. 10). In line with this, it is relevant to express ideas about mathematical structures related to pre-service teachers' teaching-related math knowledge and how this knowledge can support their students' learning. According to Mason et al. (2009), a mathematical structure is defined through "the identification of general properties which are instantiated in particular situations as relationships between elements" (p. 10). This underlines two important aspects of the definition of a mathematical structure. The first has to do with the mathematical meaning of general/essential properties, and the second with "association" between elements. These elements are mathematical objects or concepts, e.g., number concepts, algebraic concepts, and their operations. In this context, the operations are related within elements of natural numbers, whole numbers and rational numbers. In a conceptual sense, the relationship between elements is a connection between these number concepts and operations with them, as well as an extension of these operations to algebraic elements/concepts. Understanding mathematical structures as relationships between essential properties within a mathematical structure (mathematical concept), the relationship between different elements, and "mastering procedures" allows for the development of contextual meaning in an individual's mathematical knowledge. The understanding of mathematical structure can result in more effective learning focused on the re-construction of existing mathematical knowledge and the construction of new mathematical knowledge (Mason et al., 2009).

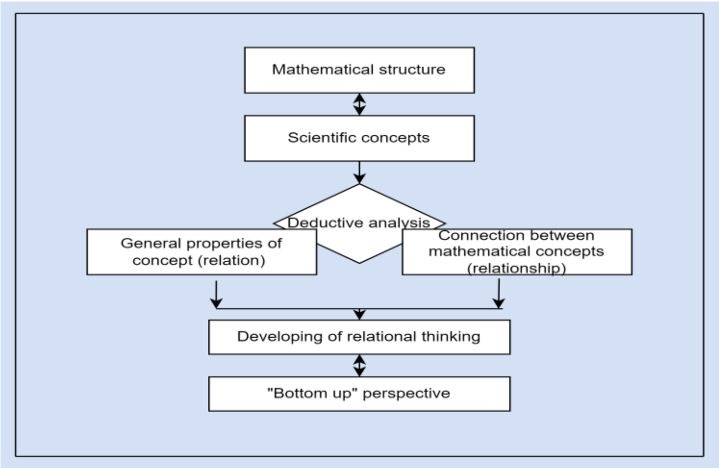
The theoretical issues surrounding mathematical structure clearly indicate which kind of mathematical knowledge makes sense in teaching and learning. In general terms "making sense" means understanding mathematical structures and the relationships between them in an arithmetic and algebraic context (Kieran, Pang, Schifter, & Fong, 2016). A mathematical structure like the Abelian group for addition describes the relationship between general/essential properties within the group and relationships between natural numbers and rules for addition, including the concept of inverse operations. The Abelian group for multiplication is structured the same way, but relates to multiplication (van der Waerden, 1971). One interpretation of the Abelian groups in a conceptual context associated with teaching is described by Karlsson and Kilborn (2024). With regard to teaching algebra, they also highlight the importance of the Abelian groups as a professionally specific form of mathematics knowledge.

Discussions about mathematical structures in instruction bring theoretical considerations about the phenomenon of mathematical thinking and contextual sense to the fore. Mathematical thinking is important for learners' analysis of mathematical structures in order to understand content, as well as for their ability to apply this knowledge to conceptual and procedural learning. Researchers have used a framework for mathematical thinking related to domains such as relational thinking (Skemp, 1976; Empson, Levi, & Carpenter, 2010; Starr, Vendetti, & Bunge, 2018), and instrumental thinking (Mason et al., 2009). Skemp (1976) describes the distinction between two different types of understanding (thinking) as relational understanding and instrumental understanding. He defines the concept of understanding as "knowing both what to do and why" (p. 2). Relational thinking/understanding encompasses such issues as what to do and why. Answering these questions demands the ability to generalize mathematical structures/concepts according to their essential properties and to make conceptual and procedural connections with other mathematical structures/concepts. Relational thinking includes understanding the properties of numbers, mathematical operations with numbers, and properties for inverse operations, as well as the ability to handle number sentences. In terms of pre-service learning for the purposes of teaching focused on relational thinking, it is important to support the conceptualization of mathematical operations, restructure operations related to transition from one number range to another and give students the skills to generalize mathematical operations from the use of natural numbers and rational numbers to algebraic symbols (Molina & Mason, 2009).

2.2. Scientific knowledge and concepts

Closer investigation of what pre-service teachers should learn demands a connection to classroom instruction and how course content ought to be taught in relation to students' learning. In the context of scientific knowledge and theoretical views about scientific (theoretical) knowledge and its significance for mathematics learning (Karpov, 2003), this study is rooted in socio-constructivism. The application of mathematical perspectives to mathematical structures is important for the development of the mathematical thinking of pre-service teachers themselves, as well as for their understanding of the same process among their students. It is also crucial to emphasize that theoretical-analytical learning effectively maximizes the opportunities for abstract thinking and the generalization of content and concepts. The question of how course content can be broken down into logical sequences learners can adapt and absorb is a key aspect of mathematical knowledge for teaching. Against this background and in terms of the mathematical preparation of pre-service teachers, content knowledge should be theorized in a manner that underscores the clear interplay between theoretical knowledge and its impact on the teaching of mathematics. In the context of socio-constructivism, learning is discussed as the generation of "spontaneous" and "scientific" concepts (Karpov, 2003). Spontaneous concepts result from the generalization of everyday experience and knowledge, and scientific concepts are born of the generalization and systematic collection of human scientific knowledge. It is crucial to understand that spontaneous concepts are necessary conditions for the construction of scientific concepts through the analysis of differences and similarities. The theoretical approach of this study focuses on how scientific knowledge and concepts lead to reflections about why pre-service teachers should learn theoretical concepts as a basis for teaching math (Bakirov & Turgunbaev, 2019). The theoretical outlines for the study are presented in Figure 1.

Figure 1. Systemized theoretical framework: Learning for teaching.



3. Teaching fractions and inverse operations

This study examines the mathematical structure of the Abelian groups for addition and multiplication, with a focus on inverse operations (van der Waerden, 1971). This is crucial content for sequence analysis related to the purpose of the study, namely analysis of how the mathematical structure of inverse operations and rational numbers can be used as a bottom-up approach in how pre-service teachers learn to teach. The analysis follows the system presented in Figure 1. The inverse operations first applied to whole numbers can be extended and applied to rational numbers. This general (essential) property of inverse operations illustrates a conceptual continuity and relationship between mathematical concepts. To illustrate this conceptual continuity, analytical findings related to both whole numbers and rational numbers are presented.

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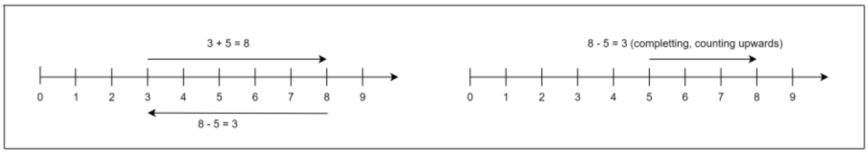
Finding 1. Addition and its inverse subtraction of whole numbers

In the Abelian group for addition, the definition of "inverse" is that for every a there is an element $(-a)$ in the group, such that $a + (-a) = (-a) + a = 0$. Here, $(-a)$ is called the (additive) inverse of a and vice versa (van der Waerden, 1971). For example, the natural number 5 is the additive inverse of (-5), and the sum of the numbers 5 and (-5) is zero. On the number line, this can be expressed as mirroring around the neutral, central element of zero. This illustrates how negative numbers are inverses of their positive, natural counterparts.

On the number line, adding 5 can be expressed as an arrow five units long pointing to the right, and subtracting 5 can be an arrow five units long pointing to the left. The addition 3 + 5 = 8 and the subtraction 8 – 5 = 3 are illustrated on the number line in Figure 2.

Figure 2. Addition and inverse operation subtraction.



The subtraction 8 – 5 = x can also be carried out as the inverse of the open addition 5 + x = 8 by *completing the number line,* i.e., counting up from 5 to 8 in 3 steps. This is shown on the number line in Figure 2 (counting upwards). If one applies the rule that the sum of two whole numbers, $b + (-b) = 0$, it is easy to understand how to carry out the subtraction $a - (-b)$. We start with the subtraction $a - b$ and then add $(b + (-b))$. By using the commutative law for addition, we find that $a - b = a + (-b)$. This means that $a - b = a +$ (the inverse of b). However, as this is true for all whole numbers, it also means that $a - (-b)$ equals $a +$ (the inverse of $(-b) = a + b$. The relational thinking about the addition of whole numbers and their inverse subtraction can be extended to rational numbers.

Finding 2. Multiplication and its inverse division of whole numbers

To examine what this means by multiplying two whole numbers, we can start with $a \cdot (-b)$, where $a > 0$. When we apply the distributive law, we find that $a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = 0$. This means that $a \cdot b$ and $a \cdot (-b)$ are inverse numbers and consequently that $a \cdot (-b) = (-a \cdot b)$. In the same way, $(-a) \cdot b = (-a \cdot b)$. How do we interpret the multiplication $(-a) \cdot (-b)$ of two negative numbers? By once again using the distributive law, we find that $(-a) \cdot b + (-a) \cdot (-b) = (-a) \cdot (b + (-b)) = 0$, which means that $(-a) \cdot (-b)$ is the inverse of $(-a) \cdot b$, whose inverse is $a \cdot b$. Consequently, $(-a) \cdot (-b) = a \cdot b$. In sum, the finding is that the product of two negative numbers is a positive number.

Finding 3. Rational numbers and inverse operations

- *Case 1*

In the Abelian group for multiplication, the definition of "inverse" is that for every a in the group (provided that $a \neq 0$), there is an element $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = 1$. In this case, $\frac{1}{a}$ is the (multiplicative) inverse of a and vice versa (van der Waerden, 1971). This means that if writing a as $\frac{a}{1}$, the inverse can be likened, metaphorically speaking, to mirroring $\frac{a}{1}$ vertically along the fractional line. If we stick with this metaphor, it is also easy to find the inverse of $\frac{a}{b}$ as $\frac{b}{a}$ and determine that $\frac{a}{b} \cdot \frac{b}{a} = 1$. In the next step, it is important to define the inverse operation for division in a way that can also be applied to the division of rational numbers. Such a definition is that $a \div b = a \cdot \frac{1}{b}$. This means, e.g., that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, and more explicitly that $\frac{2}{5} \div \frac{1}{2} = \frac{2}{5} \cdot 2$ and $\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \cdot \frac{7}{3}$. Against this background, students with only basic knowledge of algebra can understand the formula for the division of two fractions.

This is an interesting point for relational thinking. $\frac{a}{2} \div \frac{a}{2} = 1$ and $\frac{2}{a} \cdot \frac{a}{2} = 1$ independent of a; it just depends on the conceptual meaning of the division as result of an inverse operation multiplication.

- *Case 2*

Now only the multiplication of two rational numbers remains. To begin with, let us multiply two basic fractions, $\frac{1}{a}$ and $\frac{1}{b}$, and then multiply their product by $(a \cdot b)$. When we then apply the commutative and associative laws, we get $(a \cdot b) \cdot (\frac{1}{a} \cdot \frac{1}{b}) = (a \cdot \frac{1}{a}) \cdot (b \cdot \frac{1}{b}) = 1 \cdot 1$. Since the product $(a \cdot b) \cdot (\frac{1}{a} \cdot \frac{1}{b}) = 1$, we know that the inverse of $(\frac{1}{a} \cdot \frac{1}{b})$ is $(a \cdot b)$. However, the inverse of $a \cdot b$ is $\frac{1}{a \cdot b}$. Consequently, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$.

After this, it is simple to multiply two arbitrarily chosen fractions: $\frac{a}{b} \cdot \frac{c}{d}$. By first splitting things up as $a \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d}$ and then using the communicative law för multiplication, one finds that $(a \cdot b) \cdot (\frac{1}{b} \cdot \frac{1}{d}) =$ $(a \cdot b) \cdot \frac{1}{b \cdot d} = \frac{a \cdot b}{b \cdot d}$. This illustrates how relational thinking can be used to analyze the multiplication of two rational numbers.

4. Conclusions

This paper shows that mathematical structures and the concept of inverse operations of rational numbers can give pre-service teachers conceptual support as they learn how to teach math. At the same time, this analysis by theoretical framework (Figure 1) can provide a clearer understanding for pre-service teachers about how the methods of working from the bottom up and breaking mathematical concepts down as two different approaches (inductive and deductive) for teaching and learning. The analytical findings are illustrated with a clear connection to the mathematical structures/scientific concepts of inverse operations for addition and multiplication. These concepts can support student teachers' perceptions of mathematical operations and how to learn and teach them. This theoretical approach and the analytical findings of the study are currently being implemented in the teacher education programs for Grades K-3 and 4-6 at Södertörn University in Sweden.

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