https://doi.org/10.36315/2025v1end001
Education and New Developments 2025

STUDENTS' SYSTEMATIC REPETITION AS A FRAMEWORK FOR LEARNING ALGEBRA: A CASE STUDY FROM TEACHERS' PROFESSIONAL DEVELOPMENT

Natalia Karlsson[1], & Wiggo Kilborn[2]
[1]Department of Pedagogy and Didactics, Södertörn University, Huddinge, Stockholm (Sweden)
[2]Faculty of Education, University of Gothenburg, Gothenburg (Sweden)

Abstract

The purpose of this study is to examine how students' repetition of arithmetic concepts is organized during the first year of upper secondary school. The theoretical approach is a triangulation of Freudenthal's didactical phenomenology of mathematical structures, Bernstein's repetition -without -repetition and Mezirow's transformative learning theory. The study includes 94 upper secondary school students and their three mathematics teachers. The crucial content for repetition was whole numbers, rational numbers and potencies as prior knowledge for learning algebra-specific content, such as algebraic expressions, equations and problem solving. The study is part of a research project about teachers' professional development in a "research circle." The key components of the methodology are a qualitative analysis of the research questions and theoretical triangulation, which aim to identify significant empirical findings. The results of the study show that limited space is given to conceptual repetition to support students' learning of algebra. Moreover, the current textbook was not a useful source of support for the teachers.

Keywords: *Systematic repetition, prior knowledge and algebra, knowledge transfer from primary to upper secondary school.*

1. Introduction

Students' development of knowledge from the senior year of primary school to secondary school is perceived as problematic and challenging (Jindal-Snape, Symonds, Hannah & Barlow, 2021). One reason for this is that students often come from different learning environments, bringing with them different levels of knowledge and different perceptions of mathematics (Symonds & Galton, 2014). Moreover, during primary school, students often learn mathematics in different ways, using different methods and relying on the rote learning of formulas. At the same time, secondary school activities are mostly dominated by lower-level cognitive learning, rote knowledge, and numeracy (Carmichael, 2015). This may cause problems when students learn algebra which, in turn, might have consequences for their further studies in mathematics. To resolve this, it is necessary to activate and systematize students' previous learning through optimal repetition. According to Brinkmann (2012), repetition is a teaching strategy that aims to actualize the knowledge that the students already have or have had in the past.

2. Background

Before introducing new teaching content, it is important to ensure that students have adequate prior knowledge to assimilate new mathematical concepts or methods. For some students it may be enough to recapitulate and systematize what they have already learned and understood in their previous learning of mathematics (Kania, Fitriani, & Bonyah, 2023). For other students, the knowledge gap and discrepancy between prior knowledge and new knowledge that must be assimilated is sometimes extensive and require a new approach to teaching that comprises two interwoven parts: a core of prior knowledge and new knowledge. Helping students to continuously gain access to current mathematical concepts and methods is crucial for teaching in a democratic classroom (Stemhagen & Henney, 2021).

It is not sufficient for students to work with mathematical tasks they have never understood. A study of students' learning of operations with rational numbers in grades 7–9 showed major deficiencies in grade 7 regarding operations with rational numbers (Baldemir, Tutak, Nayiroglu, & Suzen, 2023). Moreover, several research studies illustrate that students face challenges in learning rational numbers

(Avcu, 2024). Several of these challenges were still present in grade 9, where most students had poor knowledge of fractions, related concepts and also lacked conceptual knowledge. Instead, they relied on formulas they were unable to manage and often got mixed up (Joutsenlahti & Perkkilä, 2024). This meant that students had limited opportunities for knowledge transfer from primary to secondary school and consequently had difficulties developing new conceptual mathematical knowledge. In other words, continuous repetition in the form of teachers' intervention in the teaching process is required in order to coordinate the students' ability to assimilate previous knowledge and a framework for new knowledge. Knowledge transfer, specifically from primary school to upper secondary school and from arithmetic to algebra, is still the subject of research (Matiti, 2022).

The purpose of this study is to examine how students' repetition of arithmetic is organized in teaching during the first year of upper secondary school. More specifically, the study seeks to address the following research questions:

RQ1: How do teachers plan and implement the repetition of arithmetic concepts?

RQ2: How do teachers assess students' prior knowledge and what support do they get from the current textbook? The study is a part of a research project about teachers' professional development in a "research circle."

3. Theoretical conceptual framework

The theoretical conceptual framework consists of triangulation of Freudenthal's (1973; 1983) didactical phenomenology of mathematical structures, Bernstein's (1996) repetition -without -repetition, and Mezirow's (2012) transformative learning theory. The triangulation is used for the conceptualization of systematic repetition in teaching and for responding to research questions RQ1 and RQ2 through the analysis of the collected data for the current research.

The importance of understanding mathematical concepts and their conceptual meaning and context is described by Freudenthal (1983). According to a fundamental didactical phenomenological analysis of mathematical concepts and structures, it is possible to define mathematics as a science and understand how to use and apply mathematics, not least in teaching. Understanding the phenomenology of "mathematical structures, knowledge of mathematics and its applications" through didactical phenomenology provides knowledge for instruction and how to follow up students' cognitive development and growth. According to Freudenthal (1983), mathematics cannot be taught without continuity of students' learning and activating their previous experience of mathematics. At the same time, mathematical instruction needs to include realistic mathematical activities (Freudenthal, 1991; Gravemeijer, 1994) that include pre-requisites and conditions for students learning mathematics in terms of possibilities and limitations. A crucial aspect of Freundenthal (1983) in terms of teaching practice is that learning is an individual process, and a student's achievements need to be analyzed in relation to the individual development and progression of knowledge rather than the outcomes of a curriculum.

The significant impact of algebra is a result of the conceptual expansion of the arithmetic and "intuitive" methods that are mostly used by students. Teaching elementary arithmetic as numbers and their operations is key to practicing algebra using algebraic rules methods (Freudenthal, 1973). At the same time, mastering algebra is on another level of knowledge and requires new learning of algebraic methods because "traditional methods from elementary arithmetic" is not enough. According to Freudenthal (1973) arithmetic and algebra are fundamentally different because arithmetic is "intuitive and close to reality or at least it should be so" (p. 287). However, algebra is abstract and grounded in "formal-symbolic methods". To minimize the conceptual difference between arithmetic and algebraic methods it is important to introduce and apply algebraic methods at an early stage when students are introduced to algebraic calculus and algebraic symbols. This didactical point can be used, for example, in students learning of algebra through illustrations of how algebraic methods are used in the arithmetic that they learnt in previous grades. In other words, this suggests that there is a need to create a bridge between prior knowledge and new knowledge, as well as make provision for conceptual repetition. This means, that the learning of arithmetic in early grades can be organized by creating a conceptual continuity from whole numbers to rational numbers, and so on.

Bernstein's (1996) conceptualization of the concept of "repetition -without -repetition" highlights that a repetition process needs to implement by every "movement" a new learning component by action. According to Bernstein, repeated practice leads to new ways of thinking. The conceptual framework for repetition is knowledge moving forwards. Bernstein adds that "absolute repetition" (one-by-one) cannot be part of a learning process that aims to successively develop knowledge. This theoretical conceptual framework is used in this study to define a conceptual meaning of repetition as a teaching phenomenon.

From the perspective of transformative learning, it is important to include students' previous knowledge in teaching (Mezirow, 2012). Learning occurs through the integration of prior knowledge into the different areas of new knowledge. According to Mezirow, the core of learning, which he calls "learning meaning schemes", is an interplay between teaching content and students' cognitive development, as a result of the relationship between prior knowledge and new knowledge. Transformative knowledge through reflections and knowledge transformation takes place through systematic repetition (Brinkmann, 2012). Brinkmann suggested a model of repetition in terms of Mezirow's starting point for learning a process as "movement through time" and claims that new knowledge is a "moving of the repetition through the time".

4. Research methodology

The study was conducted by three upper secondary school mathematics teachers and their three classes comprising 94 students (34 students in class a, 34 in class b and 26 in class c). The collected data comprised the teaching plan for repetition, transcribed discussions, the teachers' arithmetic repetition test, the teachers' analysis and systematization of the test results, the teachers' tasks for the repetition of arithmetic concepts, tasks for teaching algebra, and tasks from the current textbook (Szabo et al., 2021).

The crucial arithmetic content for repetition was whole numbers, rational numbers and potencies and the current teaching content for algebra was algebraic expressions and linear and rational equations. The main component of arithmetic repetition and algebra teaching is Freudenthal's (1973) didactical phenomenology of whole and rational numbers, as well as his point about the number concept, applied arithmetic, and the development of the number concept – the algebraic method. A theoretical point of conceptualization of inverse operations and algebraic concepts was also used (Karlsson & Kilborn, 2024), as well as the mathematics didactics structure for the teaching of arithmetic and algebra (Karlsson & Kilborn, 2015).

In relation to the present study, the key components of the methodology are a qualitative analysis of the research questions and theoretical triangulation, which aimed to identify significant empirical findings. The quantitative analysis was used to analyze the teachers' test for repetition and arithmetic tasks for repetition in the current textbook.

5. Results

With the aim of answering research questions RQ1 and RQ2, the collected data were categorized into four themes, called episodes:

Episode 1. Repetition and the teaching plan for algebra

The teachers' plan showed that the repetition of arithmetic was an isolated act, not integrated with the teaching of algebra. The repetition lasted for five weeks: one week on whole numbers, one week on fractions, one week on decimal numbers and two weeks on potencies. At the end of the five weeks, the students were given an arithmetic repetition test. The teaching plan for repetition was from the first chapter of the current textbook and it did not have any learning goals, specific mathematical tasks, or any documentation of previous teaching.

Episode 2. Repetition and the arithmetic repetition test

The test comprised 10 arithmetic tasks. Tasks 1, 2 and 3 dealt with whole numbers, task 4 with rational numbers and tasks 5 to –10 with potencies. The structure and content of the repetition tasks in the test recognized in greater detail that these tasks had an unclear goal regarding the repetition of arithmetic concepts. Moreover, there was limited conceptual context for the repetition tasks, for example, there were few tasks about the subtraction of negative numbers, subtraction and division of rational number, etc. It was also unclear how these tasks could support the students' systematization of arithmetic and further learning in algebra.

Episode 3. Repetition and students' prior knowledge of arithmetic

It was unclear how the students' prior knowledge of arithmetic would be tied to their learning of algebra. There was a clear intention on the part of the teachers to systematize the students' prior knowledge through repetition, but the teachers' strategies when working with repetition were rather intuitive, not based on knowledge of the students' learning of algebra. Thus, it was obvious that the teachers had episodic and vague perceptions of students' learning of arithmetic and knowledge about how the students' arithmetic repetition could be organized and used as a background for their learning of algebra.

Episode 4. Repetition and textbook's repetition discourse

The current textbook has a special chapter comprising 43 pages for the repetition of arithmetic tasks, aimed at a systematic repetition of prior knowledge before learning algebra. The subheadings are as follow: Whole numbers, numbers in fractional form, numbers in decimals and potencies. The number of tasks intended for this rehearsal are shown in Table 1. The subheadings, levels 1, 2 and 3 in the current textbook show the complexity of the tasks.

Table 1. Number of tasks in the textbook for repetition.

Repetition of arithmetic tasks	Level 1	Level 2	Level 3
Whole numbers	36	8	0
Fractions	56	17	7
Decimal numbers	25	10	2
Potencies	101	50	22

The quantitative mapping of the repetition tasks for arithmetic shows that the number of tasks about whole numbers, fractions and decimal number are few compared to the number of tasks about potencies The same tendency was found in the teachers' test given after the repetition. Greater emphasis has been placed on the Level 1 arithmetic task than on an equal distribution between levels and difficulty making progress through the tasks. The crucial arithmetic concepts for learning algebra as whole numbers, fractions and decimal numbers are very limited. A qualitative analysis of the tasks shows that most of the instructions for repetition describe *how* to accomplish the tasks, not *why* they should be accomplished. As an example, there is no definition of fractions like $\frac{a}{b} = a \cdot \frac{1}{b}$, where $b \neq 0$, or that $\frac{1}{b}$ is the inverse of b. Thus, it is not possible to conceptually understand the formulas for the multiplication or division of two fractions. In fact, many students perceive $\frac{a}{b}$ as division. The arithmetic tasks in the current textbook have no clear and logical structure from level 1 to level 3. For example (level 1, Fractions p. 14) includes 10 arithmetic tasks about "shortening and lengthening". The level 2 tasks about fractions comprise one task about the lowest common denominator and two tasks about the comparison of fractions. Level 3 (p. 14) includes one task about the lowest common denominator. There is no logic to the content of the structure of the repetition tasks, for example, a comparison of fractions is used to identify the lowest common denominator and should be placed at the level 1. This is a poor conceptual transfer from one level to another and there are similar tasks on level 1 in the example above.

6. Conclusions

The results of the study show that arithmetic repetition is used as a tool in teaching, but that provision has not been made for a systematic repetition of content, either in teaching or in the current textbook. Meaningful repetition cannot take place in teaching without a clear choice and the teaching content being prioritized, as well as an understanding of what should be the focus of the repetition and systematization of student's prior knowledge. Teachers need to be taught how repetition can be compatible with their teaching aims and goals, and a structured plan and sequencing of the teaching content. Regarding the present study, the design of conceptual repetition and the repetition of arithmetic tasks is based on Bernstein's and Mezirow's theoretical conceptual framework. The current textbook repetition tasks were interpreted in line with Freudenthal's perspectives on the arithmetic and algebraic content in teaching, as well as conceptualization of arithmetic operations and algebraic concepts (Karlsson & Kilborn, 2015; 2024). The results of this study describe how students' repetition of arithmetic was organized in teaching during the first year of upper secondary school.

The study shows that limited space is given to conceptual repetition that could support students' learning of algebra. Moreover, the teaching plan, repetition test and the current textbook repetition tasks have no conceptual structure and content. The results of the study cannot be generalized due to objective limitations. Regarding the reliability and validity of the current research, this has been ensured by the systematic analysis of the collected data related to the theoretical framework and the support of previous studies that indicated that students have limited opportunities to transfer knowledge from primary to secondary school. One interesting conclusion of the study is the teachers' slowness to accommodate new ideas for teaching. It is easier trying to assimilate, that is, to include new ideas in an older, non-functioning context. This often results in intellectual conflicts and a reluctance to change methods. It took several weeks for the teachers to be convinced of the benefits of changing their teaching strategies.

The researchers have a special interest in continuing this research in order to support teachers and their teaching and organizing of students' repetition through a mathematical intervention study, as well as by analyzing the impact of developing students' learning. The conceptualization of repetition phenomena with a focus on a transition from arithmetic to algebra can be useful in future research and research discussions about students' knowledge transfer through the conceptual interplay between arithmetic and algebra.

References

Avcu, R. (2024). Middle-school mathematics teachers' provision of non-examples and explanations in rational number instruction, *International Journal of Mathematical Education in Science and Technology, 55*(6), 1391-1419.

Baldemir, B., Tutak, T., Nayiroglu, B., & Suzen, A. B. (2023). 7th grade students' experiences on rational numbers identifying misconceptions, *International Online Journal of Educational Sciences, 15*(5), 900-910.

Bernstein, A. N. (1996). *Dexterity and Its Development.* Translated by Mark L. Latash. In Mark L. Latash & Michael T. Turvey (Eds.). Hillsdale, NJ: Lawrence Erlbaum Associates.

Brinkmann, M. (2012). Repetition and Transformation in Learning. A Hermeneutic and Phenomenological View on Transformative Learning Experiences. In A. Laros, T. Fuhr & E. W. Taylor (Eds.), *Transformative Learning Meets Bildung* (pp. 73-83). Sense Publishers.

Carmichael, C. (2015). Transitioning to secondary school: The case of mathematics. *Australian Journal of Educational and Developmental Psychology, 15*, 13-23.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Netherlands: D. Reidel Publishing Company.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Netherlands: D. Reidel Publishing Company.

Freudenthal, H. (1991). *Revisiting Mathematics Education: China Lectures*. Dordrecht, The Netherlands: Kluwer.

Gravemeijer, K.P.E. (1994). *Developing Realistic Mathematics Education* (Doctoral dissertation, Utrecht University, Utrecht).

Jindal-Snape, D., J. E., Symonds, Hannah, E. F. S., & Barlow, W. (2021). Conceptualizing Primary-Secondary School Transitions: A Systematic Mapping Review of Worldviews, Theories and Frameworks. *Frontiers in Education 6,* 540027. doi: 10.3389/feduc.2021.540027

Joutsenlahti, J., & Perkkilä, P. (2024). Mastery of the Concept of Percentage and Its Representations in Finnish comprehensive school grades 7-9. *Education Sciences, MDPI,* 14(10), 1043. https://doi.org/10.3390/educsci14101043

Karlsson, N., & Kilborn, W. (2015). *Matematikdidaktik i praktiken: Att undervisa i årskurs 1–6* [Mathematics didactics in practice: Teaching in years 1–6], Malmö, Sweden: Gleerups Utbildning AB.

Karlsson, N., & Kilborn, W. (2024). Teaching fractions and the concept of inverse operations: Scientific concept in pre-service teachers' learning of mathematics for teaching purposes. In M. Carmo (Ed.), *Education and New Developments 2024* (Vol.1, pp. 71-75). Lisbon, Portugal: inSciencePress.

Kania, N., Fitriani, C., & Bonyah, E. (2023). Analysis of Students' Critical Thinking Skills Based on Prior Knowledge Mathematics. *International Journal of Contemporary Studies in Education, 2*(1), 49-58.

Matiti, J. (2022). Students transitioning from primary to secondarymathematics learning: a study combining critical pedagogy, living theory and participatory action research. *Educational Action Research, 32*(1), 144-160.

Mezirow, J. (2012). Learning to think like an adult: Core concepts of transformative theory. In E.W. Taylor and P. Cranton (Eds.), *The handbook of transformative learning: Theory, research, and practice* (pp.73-95). San Francisco, CA: Jossey-Bass.

Stemhagen, K., & Henney, C. (2021). *Democracy and mathematics education. Rethinking school math for our troubled times.* New York, NY: Routledge, Taylor & Francis Group.

Symonds, J., and Galton, M. (2014). Moving to the next school at age 10–14 years: an international review of psychological development at school transition. *Review of Education, 2*(1), 1-27. doi:10.3389/fpsyg.2018.01482

Szabo, A., Larson, D., Dufåker, D., Fermsjö, R., Viklund, G., & Marklund, M. (2021). *Matematik Origo 1b* [Mathematics Origo 1b], Stockholm: Sanoma Utbildning AB.