

RETHINKING ALGEBRA LEARNING BASED ON A PSYCHOLOGICAL AND TRANSFORMATIVE LEARNING APPROACH

Natalia Karlsson¹, & Wiggo Kilborn²

¹Department of Pedagogy and Didactics, Södertörn University, Huddinge, Stockholm (Sweden)

²Faculty of Education, University of Gothenburg, Gothenburg (Sweden)

Abstract

The objective of this study is to examine how psychological approaches, such as transformative learning, can contribute to rethinking students' algebra learning and enhancing algebra instruction in primary and secondary school settings. The study explores conceptual continuity in students' algebra learning, as well as the interplay between number concepts, with a focus on learning whole and rational numbers and algebraic methods. It also explores the conceptual relationship between numbers and their operations, particularly the shift from numerical reasoning to abstract algebraic content through the integration of intuitive inductive algebraic methods and algebraic inverse methods. To provide background for the teaching and learning of algebra, a theoretically grounded design for teaching and learning algebra was constructed, incorporating transformative learning (TL) into algebra learning content in the form of a Transformative Structured Learning (TSL) model. The TSL model is a theoretical conceptualization of how the learning process should engage, systematize, extend, and conceptually integrate students' prior knowledge (experience) into a new algebraic context. The benefit of the TSL model is that it illustrates how a conceptual transition from students' prior knowledge to algebraic content can be built and applied to other types of teaching–learning contexts, such as instrumental, dialogic-communicative, and self-reflective, focusing on arithmetic–algebra content. The algebraic methods in the TSL model function as mediators and conceptual transition tools that lead students to a deeper transformative understanding of algebraic sign rules and algebraic operations with numbers. The paper outlines a new, theoretically grounded approach to teaching and learning algebra based on a TL learning design, such as the TSL model, which is adjusted according to the subject-specific environment of individual classrooms. The study highlights the importance of developing alternative approaches to teaching and learning – focusing on the psychological nature of learning – thereby supporting students and their understanding of algebra.

Keywords: *Learning of algebra, transformative learning, conceptual continuity, whole and rational numbers, algebraic inverse concept, Freudenthal's inductive method.*

1. Introduction

Several researchers emphasize that learning algebra is a difficult process for students as they transition from arithmetic to abstract algebra (Sowder, Sowder, & Nickerson, 2010; Holmlund, 2025). According to foundational research focusing on Freudenthal's (1973) didactical phenomenology of teaching and learning arithmetic and algebra, "the first exercises in writing and arithmetic are closely connected with each other" (p. 1). However, abstract algebra has a more symbolic, stringent, and logical nature. Freudenthal also argues that any teaching process that lacks continuity (i.e., "an isolated subject") is not a successful learning process and that learning occurs when students are able to generalize solutions and understand them more profoundly (p. 95). The key to unlocking this kind of learning is algebraic knowledge.

In addition, if students demonstrate prior arithmetic knowledge solely through repetition, without incorporating new algebraic concepts, they are likely to learn algebraic content in isolation. This approach does not account for new knowledge and does not prepare students for advanced mathematics, including algebra. Continuity in teaching and learning is therefore an essential component of students' cognitive development (Vygotsky, 1987). Algebraic concepts and methods are usually introduced to students through algebraic expressions and linear equations in one variable. However, the gap between numerical and algebraic concepts and methods prevents students from assimilating a perspective of conceptual continuity (Kieran, 2004). From a learning perspective, algebraic concepts and actions should be mediated early within

the framework of arithmetic learning (Karlsson & Kilborn, 2023). Learning tools related to algebraic rules—such as algebraic sign rules, rules for algebraic operations, first and second squaring rules, the conjugate rule, basic laws of algebra, and inverse operations—when introduced early in mathematics instruction through activity, support students’ understanding of symbolic systems and algebraic language, as well as the interplay between arithmetic and abstract algebra (Karlsson & Kilborn, 2024).

The paradigm shift from arithmetic to algebraic concepts and methods includes students’ understanding of algebraic letters and the transition to the language of algebraic formulas, which are considered unfamiliar and particularly challenging for students’ learning (Fülöp, 2015). Several researchers emphasize that this transition is difficult to achieve (e.g. Wahyuni, Herma, & Fatimah, 2023). The algebraic letter functions as part of a symbolic system and requires mastery of the system’s properties through continuous learning. Numerous theoretical and empirical studies have examined students’ interpretations of algebraic letters (e.g., unknowns and variables) (e.g., Qin & Opfer, 2025).

There is still limited theoretical and empirical research on which types of content facilitate a conceptual transition from arithmetic to algebraic knowledge, particularly regarding the generalization of algebra through language-oriented, dialogic verbalization, as well as the development of algebraic learning from arithmetic to algebraic paradigm shifts and how these processes are addressed in the teaching and learning of algebra (Fajardo, Gonzalez, & Urbano-Leon, 2025). The development of algebraic language is a critical factor, and misunderstandings of this language often lead to difficulties in the learning of algebra. Oral verbalization fluency, to a level appropriate for students, is necessary for effective algebra learning (Kinard & Kozulin, 2003). From this perspective, the most problematic aspect is that algebraic language cannot be practiced outside the classroom and “is only rarely exercised” within the teaching and learning context. The aspects described above form the conceptual foundation for the present study.

2. Purpose

The purpose of this analytical study is to investigate how the concept of transformative learning (TL) (Mezirow, 2012), as a psychological approach, can contribute to understanding how students learn algebra. The study also examines a theoretically grounded Transformative Structured Learning (TSL) model, which facilitates an optimal interplay between students’ prior knowledge and new algebraic content, focusing on the conceptual relationship between arithmetic and algebraic content.

2.1. Study objective

The objective of this study is an explicit aim of the current analytical investigation and is intended to contribute to the development of a Transformative Structured Learning (TSL) model for algebra. This model represents a methodologically rigorous approach based on an interdisciplinary theoretical framework, drawing on both mathematics education and psychology.

The analytical investigation presented in this study is a logical continuation of previous analytical and empirical research on students’ algebra learning (Karlsson & Kilborn, 2023, 2024, 2025) and provides an alternative teaching and learning design that incorporates transformative learning to address algebraic content.

3. Theoretical framework

3.1. Arithmetic and algebraic conceptual learning field

In accordance with Freudenthal’s (1973) perspective, arithmetic constitutes an important part of students’ learning, but there is a conceptual gap between arithmetic and algebra. From this theoretical standpoint, arithmetic is intuitive and connected to real-world situations, “or it should be so” (p. 287). In contrast, algebra is abstract and involves “formal-symbol methods.” The use of commonly adopted algebraic methods in teaching and learning arithmetic is a crucial factor in the development of students’ algebraic skills (Freudenthal, 1973). In line with this, the development of number concepts (e.g., whole and rational numbers) occurs through a process comprising several phases, in which operations with numbers and algebraic operations play a central role (p. 242). Within this conceptual context, the organization of structural addition and subtraction, as well as multiplication and division, is highlighted. A key aspect is the integration of whole and rational numbers with *intuitive algebraic methods* (inductive–extrapolatory methods) and *algebraic inverse methods*.

3.1.1. Intuitive inductive algebraic methods. The intuitive algebraic methods serve as a didactical resource that can illustrate, for example, negative numbers (Freudenthal, 1991). In learning algebraic methods, such as algebraic sign rules, understanding how positive (+) and negative (–) numbers interact in addition and subtraction requires the rationalization of intuitive arithmetic thinking. Within this context,

negative numbers and their operations (addition, subtraction, and multiplication) can be understood intuitively through an inductive–extrapolatory method (see Figure 1) (Freudenthal, 1973).

Figure 1. Freudenthal's inductive–extrapolatory methods.

$3 + 2 = 5$	$3 - 2 = 1$	$3 \cdot 2 = 6$	$(-3) \cdot 2 = -6$
$3 + 1 = 4$	$3 - 1 = 2$	$3 \cdot 1 = 3$	$(-3) \cdot 1 = -3$
$3 + 0 = 3$	$3 - 0 = 3$	$3 \cdot 0 = 0$	$(-3) \cdot 0 = 0$
$3 + (-1) = \dots$	$3 - (-1) = \dots$	$3 \cdot (-1) = \dots$	$(-3) \cdot (-1) = \dots$
$3 + (-2) = \dots$	$3 - (-2) = \dots$	$3 \cdot (-2) = \dots$	$(-3) \cdot (-2) = \dots$

The formalization of induction illustrated in Figure 1 provides an intuitive interpretation of algebraic sign rules. This mapping demonstrates to students the potential for developing intuitive arithmetic knowledge related to algebraic rules based on an inductive thinking method.

3.1.2. Algebraic inverse methods. Algebraic inverse methods, such as the inverse operation, illustrate the transition from natural numbers to negative numbers and from whole numbers to rational numbers (Vergnaud, 1994). This organization of number expansion through inverse operations helps students understand how to extend the number system using algebraic inverse methods (Karlsson & Kilborn, 2025) and facilitates the progression from known to unknown knowledge. In other words, students have the opportunity to further develop their prior arithmetic knowledge within a new conceptual framework. At the same time, algebraic inverse methods, as an algebraic concept (van der Waerden, 1971), provide a learning activity in which students engage and develop both arithmetic and algebraic thinking (Freudenthal, 1973, p. 279).

The inverse concept is a central component of algebraic methods and supports the purposeful development of students' algebraic thinking alongside arithmetic learning activities. Inverse algebraic thinking also facilitates the understanding and application of operations and their corresponding inverse operations, such as subtraction as the inverse of addition, division as the inverse of multiplication, and exponentiation as the inverse of the square root (Karlsson & Kilborn, 2025). The algebraic concept of the inverse creates a relational conceptual field between numbers and their operations, which should be actively employed to support students' algebra-based arithmetic learning. While the bridge between arithmetic and algebra through the concept of the inverse is not sufficient for mastering abstract algebraic concepts, it provides an optimal conceptual knowledge base for understanding numbers and their operations and offers an effective approach to learning algebra. Introducing algebraic methods (e.g., the inverse) and their relational meaning within the arithmetic context, in combination with teaching and learning strategies that foster language-oriented, dialogic algebraic skills, is an important area of research.

4. Methodology

4.1. Transformative learning (TL) design for teaching and learning algebra

The conceptual psychological framework of transformative learning (TL) (Mezirow, 2012) provides a model for understanding the learning process across various domains, including instrumental, dialogic (communicative), and self-reflective learning. Instrumental learning concerns acquiring subject-specific facts, concepts, methods, and the relationships between them (e.g., arithmetic number concepts and the relationships between natural and negative numbers, between whole and rational numbers through algebraic inverse methods, and operations with whole numbers and the relationships between operations through intuitive inductive algebraic methods). The instrumental learning domain also facilitates students' regular use of algebraic methods in connection with arithmetic number concepts. The dialogic-communicative learning domain involves enriching students' verbal algebraic skills and algebraic language by testing their verbalization of algebraic methods and understanding how these methods are mastered. Intuitive algebraic methods guide students in identifying patterns, relationships, and structures, as illustrated in Figure 1. The close numerical analog of algebraic sign rules (Figure 1 conceptual track) activates students' arithmetic knowledge through associative memory (Marcus, 2001). From this perspective, algebraic knowledge is more effectively acquired by alternating learning activities between instrumental and dialogic processes, where instrumental facts are encoded from arithmetic into an algebraic context. In this area, teaching and learning through systematic verbalization and practice has been reported as one of the most effective approaches for developing knowledge of arithmetic and algebraic methods (Chen, Yanke, & Campbell, 2016). The quality of teaching and learning arithmetic in connection with the

algebraic conceptual field also depends on students' self-reflective learning. Systematic reflection on how students engage with learning content and how to improve performance helps them understand how and why they have learned. Self-reflective learning develops students' ability to analyze, systematize, and build on prior knowledge and fosters, through "thoughtful action," the growth of independent understanding and intellectual development (Taylor, 2000).

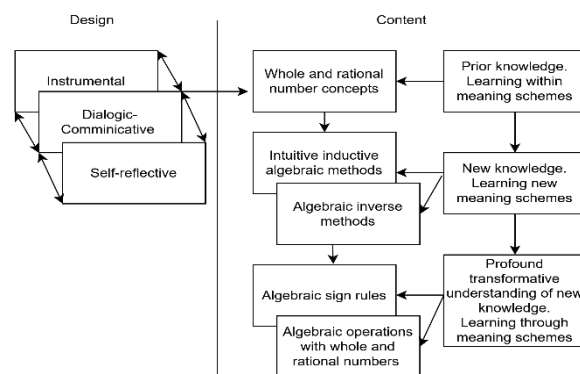
Another key aspect of TL (Mezirow, 2012) is its provision of a logical structure for the learning process, illustrating how students can master knowledge as well as subject-specific content. The framework consists of three interdependent learning phases, referred to as meaning schemes, which describe how individual learning skills and experiences contribute to the development of knowledge (Mezirow & Taylor, 2009). These phases are *learning within meaning schemes*, *learning new meaning schemes*, and *learning through meaning schemes*. The learning within meaning schemes indicates that the learning process begins with prior knowledge, which must be systematized and integrated into the learning of new material, corresponding to the learning new meaning schemes phase. The conceptual relationship between prior knowledge and new knowledge is a critical condition for effective learning. In the context of this study, this involves the integration of arithmetic and algebraic methods, as well as the connection between number concepts and algebraic methods, such as intuitive inductive and inverse methods. The TL theoretical framework provides insight into students' capacity to learn, apply, and generalize previously acquired skills to new concepts, such as algebra, based on their existing arithmetic knowledge.

In terms of advanced teaching and learning of arithmetic concepts, the implementation of algebraic inverse methods provides the TL approach with a key methodological component, encompassing the learning of algebraic content that grows out of arithmetic through instrumental learning activities, as well as the development of algebraic language through dialogic learning activities. Another contribution of TL to a new methodology for teaching and learning algebra is its structured learning process, which guides how students can master algebra-specific content through three interdependent phases. Learning begins with students' prior experiences; for example, natural numbers serve as the starting point for learning negative numbers, which emerge as a transition from the conceptual field of natural numbers to that of negative numbers, facilitated by the use of inverse methods (Karlsson & Kilborn, 2025).

4.2. Incorporation of TL design into algebra learning content: The Transformative Structured Learning (TSL) model

The study promotes an analytical, multidisciplinary methodological design for teaching and learning algebra, explored through a theoretically grounded Transformative Structured Learning (TSL) model, which provides an optimal interplay across interconnected algebra learning activities (Figure 2).

Figure 2. Transformative Structured Learning (TSL) model.



5. Conclusions

This study analytically explores a TSL approach that provides conceptual continuity in students' learning through the generalization and verbalization of algebraic methods, as well as the integration of number concepts with algebraic methods. It also examines the conceptual relationship between numbers and the transition from numerical reasoning to abstract algebraic content through inverse methods (van der Waerden, 1971; Vergnaud, 1994; Karlsson & Kilborn, 2025) and intuitive methods (Freudenthal, 1973, 1991).

The benefit of the design is that integrating arithmetic–algebraic content with a TL-informed approach and subject-specific arithmetic content, combined with the incorporation of algebraic methods

through inverse operations, leads to a new TSL methodology for teaching and learning. Analysis of the research field on knowledge transition from arithmetic to algebra, along with the results of this theoretical study, highlights the importance of effective approaches to teaching and learning algebra that focus on the psychological nature of learning, thereby supporting students' understanding of algebra in early learning.

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