

DISCURSIVE ALGEBRAIC SKILLS AND THEIR IMPACT ON K-3 PRE-SERVICE TEACHERS' MATHEMATICAL WRITTEN EFFICIENCY

Natalia Karlsson¹, & Wiggo Kilborn²

¹Department of Pedagogy and Didactics, Södertörn University, Huddinge, Stockholm (Sweden)

²Faculty of Education, University of Gothenburg, Gothenburg (Sweden)

Abstract

The objective of this study was to analyse the relationship between pre-service teachers' (PSTs') discursive algebraic skills and their written mathematical (specifically algebraic) achievements. The conceptual design consists of three theoretical perspectives that are merged in the study. This study used a mixed-methods design, combining quantitative and qualitative approaches. The PSTs completed an algebraic knowledge test designed to identify their discursive and algebraic knowledge, and a content analysis of their written texts was then performed. The written texts focused on algebraic patterns, concepts and their generalisation related to rational numbers, patterns in multiplication tables, proportionality and combinatorial concepts, including permutations and combinations. The results of this study are not generalisable due to objective limitations and the contextualisation of the research within a specific learning environment. However, the findings confirm the theoretical perspectives proposed by Vygotsky, Freudenthal and Kriegler's theoretical framework, and support the view that the development of prospective teachers' writing ability is not an isolated learning process. Rather, it likely occurs through interaction with the acquisition and development of algebraic content knowledge. Future research should include more empirical and analytical studies on the development of PSTs' discursive algebraic knowledge, particularly the processes underlying the formation and growth of proficiency in mathematical writing.

Keywords: *Discursive algebraic skills, algebraic concepts and generalisation, mathematical writing efficiency, pre-service teachers learning of algebra, algebraic thinking.*

1. Introduction

To establish common ground for learning algebra, several researchers focus on algebraic concepts that are challenging to teach and learn (e.g., Kusuma, Waluya, Rochmad, & Mariani, 2024). A common research finding is that the development of PSTs' algebraic skills and knowledge depends on many areas of algebraic content, including algebraic notation and subject-specific writing, where algebraic concepts and generalisation form the primary conditions for learning mathematics (Kozulin, 1998). Algebraic concepts form coherent patterns; however, if PSTs' knowledge of these concepts is fragmented, they cannot create a coherent learning environment for prospective teachers. Consequently, their future students may not develop a deeper understanding of algebra (e.g. Lui & Bonner, 2016), because the learning environment teachers create is closely tied to their own understanding of algebra and algebraic concepts. Examining how teachers link different content areas can shed light on students' opportunities to understand these connections. Algebraic formal thinking is an important part of learners' mathematical development, in which the generalisation of algebraic concepts plays a central role (Vygotsky, 1986). Algebraic generalisations of patterns and the relationships between algebraic concepts are key components of algebraic thinking, algebraic written notation and written algebra-related explanations and subject-specific texts. In alignment with this, Zazkis and Liljedahl (2002) emphasise that PSTs face challenges in expressing algebraic generalisations using algebraic notation. It follows those students, including prospective teachers, need algebraic instructional designs that support the learning of algebraic concepts, foster conceptual meaning and understanding of these concepts, and promote the development of algebraic notation. Algebraic symbolic notation provides a conceptual basis for the development of PSTs' discursive skills, such as the ability to use subject-specific language and thinking, in this case algebraic thinking. According to Vygotsky (1986), algebraic symbolic tools support the understanding of logico-mathematical thinking and the development of learning through the accumulation of cognitive, subject-oriented language, which is a key foundation for the development of discursive skills. Learners' mathematical development, in which algebraic generalisations require mastery of rigorous logical thinking both verbally and through algebraic

notation and writing, can foster the development of discursive skills and algebraic thinking processes (Radford, 2014). In this regard, it is important to recognise that PSTs need to master knowledge of algebraic concepts and develop discursive skills related to symbolic algebraic notation and subject-specific language, both verbal and written, in order to systematically develop knowledge for teaching algebra. According to Freudenthal (1973), symbolic algebra, including algebraic language, is strongly associated with abstract deductive thinking. Therefore, PSTs' algebraic knowledge and their ability to use algebraic concepts and provide meaningful explanations of these concepts verbally and through written notation during the teaching process should be taken into account in education. Students' difficulties and challenges in learning algebraic symbols and language depend on their teachers' capacity to handle algebra effectively. Discursive algebraic representations relate to subject-specific writing using words, signs, symbols and terminology to represent concepts, essential properties, algebraic operations or ideas used in the systematisation, generalisation and abstraction of algebra. Discursively, the study of these representations examines how words, images and symbols in algebra create meaning for learners, and how the relationship between a sign, a symbol and what it represents is interpreted and managed in abstract formal algebra (Duval, 2017). Recently, the interplay between PSTs' discursive algebraic skills, such as knowledge of algebraic concepts, generalisation of concepts and algebraic thinking, and their mathematical writing proficiency has garnered attention in teacher education and in the current study. The body of this research underlines the importance of PSTs' algebraic skills, particularly in relation to algebraic concepts, generalisation of concepts and algebraic thinking, and how these influence their ability to write mathematical texts with discursive components.

2. The purpose of the study

The study examined the relationship between pre-service teachers' discursive algebraic skills and mathematical writing efficiency in order to extend knowledge about PSTs' conditions for learning algebra.

3. Conceptual design and this study

The conceptual design refers to a triangulation of three conceptual perspectives: (1) Vygotsky's (1986) perspective on algebra, abstraction and generalisation; (2) Freudenthal's (1973) distinction between intuitive arithmetic thinking and algebraic abstract thinking; and (3) selected components of Kriegler's (2008) theoretical model related to algebraic thinking. This triangulation provides a conceptual basis for the formation of the study's objectives, particularly concerning discursive algebraic skills assessed through the algebra knowledge tests and written texts. This conceptual framework provides the theoretical background and has been integrated into the study.

3.1. Algebraic concepts, generalisation and abstraction

Discussions about the role played by algebra and algebraic thinking in students' learning appeared in Vygotsky's (1986) research. He stated that algebraic knowledge is a specific form of abstract knowledge and that students who have mastered algebra achieve a higher level of thought. According to Vygotsky, algebraic thinking provides a conceptual basis for generalisation and represents a progression from a 'lower' arithmetic level to abstraction, as well as from the concrete to the abstract. In order to attain the higher level of thought described by Vygotsky, students need to learn algebraic concepts expressed through arithmetic and to develop these concepts from arithmetic to algebra. This involves learning theoretical ('scientific') concepts based on semiotic systems related to signs, terminology and symbolic forms that incorporate the field of algebraic theoretical concepts. Empirical concepts built from everyday experience, referred to by Vygotsky as precepts and situated within the arithmetic domain, are not sufficient for the development of algebraic thinking. Several researchers emphasise that algebraic concepts are coherent theoretical abstractions and that students' algebraic development is built through formal thinking involving description (verbalisation), construction and definition (e.g. Kieran, 2007), which also leads to quantitative reasoning skills such as inductive and deductive reasoning (e.g. Smith & Thompson, 2007). Therefore, approaches to defining algebra and supporting the development of algebraic thinking cannot occur without algebraic concepts and their generalisation, where arithmetic concepts may support learning, but the transition from arithmetic to formal algebra requires learners to understand the distinction between arithmetic and algebraic thinking and the abstract nature of algebra, including the use of signs and symbols to describe different algebraic situations. (Derry, Wilsman, & Hackbarth, 2007).

3.2. Intuitive arithmetic thinking and abstract algebraic thinking

In consideration of Vygotsky's position on algebraic thinking, Freudenthal (1973) indicated that arithmetic and algebraic concepts are separated by a conceptual gap, and that arithmetic concepts alone

cannot fully support the development of algebraic concepts and thinking. Learning symbolic algebra and algebraic expression requires an understanding of fundamental algebraic ideas, the development of cognitive tools, and other analytical strategies. Furthermore, researchers emphasise that the introduction of algebraic concepts and their generalisation early in schooling – informally, through integration of arithmetic and algebra, with the incorporation of algebraic thinking components, and supported by verbal and discursive written learning activities (Kaput, 2000) – is a crucial element of early algebra learning.

Freudenthal's theoretical analysis emphasises the distinction between arithmetic and algebraic conceptual structures. He argued that arithmetic concepts and methods are inherently based on *intuitive* logic and thinking. In contrast, algebraic concepts, generalisation and algebraic thinking involve a different, more abstract and logical organisation, which requires learners to study algebra through this specialised relational structure of concepts and relational thinking, expressed verbally, through notation and in algebraic writing (Chimoni, Pitta-Pantazi, & Christou, 2023).

3.3. Algebraic thinking and discursive algebraic skills

The formulation of algebraic thinking and its organisation has been proposed by Kriegler (2008). This framework incorporates components designed to define the meaning of algebra learning. The components of algebraic thinking are organised into two domains: mathematical thinking tools and fundamental algebraic ideas. Mathematical thinking tools are divided into three interconnected parts: problem-solving skills, representation skills and quantitative reasoning skills. Fundamental algebraic ideas include algebra as generalised arithmetic, algebra as a language and algebra as a tool for functions and mathematical modelling. This organisation of algebraic thinking suggests that the development of algebraic skills occurs when a solid discursive ability is also linked to the understanding of algebraic concepts, patterns and generalisation.

3.4. Conceptualisation of the incorporation of theoretical triangulation

A structural map of the conceptual design for the study is linked to the conceptual framework, which combines the three theoretical perspectives described above, and focuses on identifying and analysing PSTs' discursive algebraic skills and written mathematical and algebraic achievements. The research instrument for analysing discursive algebraic skills and written mathematical achievements is structured across three levels: Level 1 (L1), Level 2 (L2) and Level 3 (L3). L1 covers arithmetic skills, including arithmetic concepts and intuitive methods. L2 encompasses representations and quantitative reasoning skills, including algebraic methods (generalisation of arithmetic concepts) and inductive reasoning. L3 addresses discursive algebraic skills, including writing explanations in words within an algebraic context, representing algebraic contexts using tables, providing explanations through algebraic tasks, using signs, symbols and terminology, and applying deductive (meta) reasoning through algebraic abstraction.

4. Methodological approach

4.1. Context, data collection and participants

This study adopted a mixed-methods design, combining quantitative and qualitative components within the context of primary school mathematics education, as part of a five-week course (Advanced Mathematics Development) for pre-service K–3 teachers. The study examined PSTs' discursive algebraic skills and their impact on written mathematical, specifically algebraic, achievements. At the beginning of the course, during the first seminar, PSTs completed a knowledge test designed to assess their algebraic knowledge. At the end of the course, PSTs produced written texts focusing on the course's algebraic learning content. The algebraic content of the course included combinatorics, with a focus on algebraic generalisations, algebraic patterns in multiplication tables, rational numbers, and their conceptual connection to algebraic generalisations related to proportionality. Data collection comprised the algebraic knowledge tests and written texts of *eighteen* PST participants. Quantitative and qualitative content analyses examined discursive patterns in accordance with the research instrument described in Section 3.4, applied to both the algebraic knowledge tests and the written texts.

4.2. Algebraic knowledge test and written text

Data were collected in two rounds through the algebraic knowledge test and then through written texts. The collected data used in the study were coded. Validity and reliability were established by the independent review of each of the researchers, citing the collected data in the theoretical context and clarifying the coding of tests and texts. The focus of the analysis is an objective phenomenon related to the purpose of the study. The PSTs completed a knowledge test designed to identify their discursive knowledge and algebraic knowledge, and a content analysis of their written texts was then performed. *Knowledge test:*

The algebraic knowledge test comprised four problem-solving tasks, of which two tasks dealt with algebraic methods and two dealt with the algebraic concept of proportionality. *Written text:* The written text has a structure consisting of three pieces (P1, P2, P3) as follows: (P1) identifying algebraic concepts in a textbook for K–3 teaching; (P2) constructing two teaching tasks focusing on an algebraic concept; and (P3) constructing how the teaching tasks (P2) can be carried out in K–3 teaching. The written texts focused on algebraic concepts related to rational numbers and proportionality, algebraic patterns, patterns in multiplication tables, and combinatorial concepts, including permutations and combinations.

4.3. Ethical considerations

The algebra tests and the written texts were coded, interpreted and analysed, focusing on Swedish ethical principles such as objectivity and anonymity. The collected data were treated confidentially, and conclusions were based on objective sources, such as subject-specific content. Conclusions focused on ethical standards and followed the Swedish guidelines for Good Research Practice (Swedish Research Council, 2024), which outline principles for research integrity and ethics.

5. Results

5.1. Algebraic knowledge test

The results of PSTs' algebraic knowledge tests are presented in Table 1.

Table 1. Results of PSTs' algebraic knowledge tests.

L1. Arithmetic skills		
Arithmetic concepts	Intuitive methods	
7(18)	7(18)	
L2. Representations and quantitative reasoning skills		
Algebraic methods (generalisation)	Inductive reasoning	
6 (18)	6(18)	
L3. Discursive algebraic skills		
Tables/Sketch	Sign, symbols and terminology	Deductive reasoning
10(18)	6(18)	1(18)

The *quantitative results* in Table 1 show PSTs' discursive algebraic skills. The results of the quantitative analysis were also categorised into four sub-result groups: SG1, SG2, SG3 and SG4. SG1 included 5 of 18 tests, showing partial incorporation of arithmetic concepts and intuitive arithmetic methods, and incorrect arithmetic solutions and answers. SG2 included 7 of 18 tests, with incorporation of arithmetic concepts, correct use of intuitive (conjectural) arithmetic methods, and correct or partially correct arithmetic solutions and answers. SG3 included 5 of 18 tests, with incorporation of elements of algebraic concepts and methods and correct or partially correct algebraic solutions and answers. SG4 included 1 of 18 tests, demonstrating correct algebraic concepts and methods, and correct algebraic solutions and answers. *Qualitative data* analysis offered additional information regarding L1, L2 and L3, including discursive skills and achievements. It can be stated that PSTs had difficulties and challenges in dealing with algebraic concepts and algebraic methods (generalisation) (groups SG1, SG2 and SG3), and some students were even unsure about arithmetic concepts and methods (groups SG1 and SG2). This made an impact on their discursive and written achievements. Most results of the algebraic tests showed that PSTs did not use correct arithmetic–algebraic notation. Most of the tests included very limited explanations using words, signs and symbols. Algebraic terminology was a challenge for all 18 students. The results show challenges for PSTs in thinking in terms of arithmetic and/or algebraic approaches.

5.2. Written text

Difficulties explaining algebraic concepts correctly. Ten of 18 participants had challenges describing the chosen algebraic concept correctly. A typical situation was that participants were asked to describe algebraic patterns in the multiplication table, such as commutativity, but instead described the concept of multiplication without a conceptual connection to the structure and meaning of the multiplication tables. *Difficulties in formulating algebraic tasks focusing on algebraic concepts.* Results reported that 7 of 18 students had difficulties formulating algebraic tasks. Most of the formulated algebraic tasks were related to everyday situations and led directly to counting. *Difficulties in explaining algebraic tasks using words, symbols, and terminology.* Ten of 18 participants had challenges with explanations of algebraic concepts and tasks, and six of 18 reported these difficulties. Some typical results were that

algebraic-specific content was described in general terms, focusing on how students might handle the concepts or methods. For example: “The student can also write an expression that shows the structure: $3 \cdot (4 \cdot 3)$. [...], the student is given the opportunity to choose the strategy that best supports their understanding.” *Difficulties with deductive (meta) reasoning (algebraic abstraction)*. Only one of the participants used deductive reasoning to explain algebraic tasks. In general, the results clearly show that PSTs’ discursive algebraic skills had a strong impact on their written efficiency.

6. Conclusions and discussion

The study examined the relationship between discursive algebraic skills and mathematical writing efficiency. Both qualitative and quantitative analyses indicated that the study is timely and that PSTs need to strengthen their knowledge and skills in fundamental algebra, particularly discursive algebraic knowledge, as a key factor in developing their professional linguistic ability, in this case, written mathematical text. The results illustrated that PSTs’ written algebraic efficiency could not develop optimally without solid algebraic knowledge. The ability to perform algebraic generalisation, use algebraic generalisation, and apply deductive reasoning is key to writing ability. This means that addressing writing problems requires careful handling and treatment through research knowledge and practical educational solutions, which is very important for future students. In conclusion, more empirical and analytical studies are needed on the development of PSTs’ discursive algebraic knowledge, particularly the processes underlying the formation and growth of efficiency in mathematical writing. *Limitations of the study are that the results are not generalisable due to objective constraints and the contextualisation of the research within a specific learning environment.* However, the findings confirm the theoretical models proposed by Vygotsky (1986), Freudenthal (1973) and Kriegler (2008) and support the view that the development of PSTs’ writing ability is not an isolated learning process. Rather, it likely occurs in interaction with the acquisition and development of algebraic content knowledge.

References

- Chimoni, M., Pitta-Pantazi, D., & Christou, C. (2023). Unfolding algebraic thinking from a cognitive perspective. *Educational Studies in Mathematics*, 114, 89–108.
- Derry, S., Wilsman, M., & Hackbarth, A. (2007). Using contrasting cases to deepen teacher understanding of algebraic thinking. *Mathematical Thinking and Learning*, 9(3), 305–329.
- Duval, R. (2017). *Understanding the mathematical way of thinking: The registers of semiotic representations*. Switzerland: Springer International Publishing AG. Springer Nature.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Netherlands: D. Reidel Publishing Company.
- Kaput, J. K. (2000). *Transforming algebra from an engine of inequity to an engine of mathematical power by “algebrafying” the K-12 curriculum*. Dartmouth, MA: National Center for Improving Student Learning and Achievement in Mathematics and Science.
- Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Charlotte, N. C.: Information Age Publishing.
- Kozulin, A. (1998). *Psychological tools: A sociocultural approach to education*. Cambridge MA: Harvard University Press.
- Kriegler, S. (2008). *Just what is algebraic thinking*. UCLA: Department of Mathematics. [Online]. Available at: <http://www.math.ucla.edu/kriegler/pub/algebrat.html>
- Kusuma, A. P., Waluya, S. B., Rochmad, R., & Mariani, S. (2024). Algebraic thinking profile of pre-service teachers in solving mathematical problems in relation to their self-efficacy. *EURASIA Journal of Mathematics, Science and Technology Education*, 20(11).
- Lui, A. M., & Bonner, S. M. (2016). Preservice and inservice teachers’ knowledge, beliefs, and instructional planning in primary school mathematics. *Teacher and Teacher Education*, 56, 1–13.
- Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal*, 26(2), 257–277.
- Smith, J. P., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York: Erlbaum.
- Swedish Research Council (2004). *Good research practice 2004*. Stockholm: Swedish Research Council.
- Vygotsky, L. S. (1986). *Thought and language* (rev. ed). Cambridge, MA: Harvard University Press.
- Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49(3), 379–402.